Dimension expanders and quiver representations

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Motivation: Expanders

An *expander* is a graph Γ which is both sparse and highly connected.
$\Gamma = (V, E)$ k-regular graph (every vertex $v \in V$ incident with k edges $e \in E$). For $V' \subset V$, define its boundary $\partial V'$ as the vertices connected to V', but not in V'. Take $\varepsilon > 0$.

Definition

A k-regular graph Γ is called an $\varepsilon\text{-expander}$, if for every $V' \subset V$ with $|V'| \leq \frac{1}{2}|V|$, we have $|\partial V'| \geq \epsilon \cdot |V|$.

For fixed (small) k and fixed (large) ε, we want to know that there exist k-regular ε-expander graphs of arbitrarily large size $|V| \to \infty$.
This concept has many applications in theoretical computer science, geometric/combinatorial group theory, hyperbolic geometry.... (see Lubotzky).

Dimension expanders – definition

F algebraically closed field, V finite dimensional F-vector space, $T_1, \ldots, T_k \in \mathrm{End}(V)$ linear operators, $\varepsilon > 0$.

Definition

$(V, T_1, \ldots, T_k)$ is called an ε-expander if, for *all* $U \subset V$ such that $\dim U \leq \frac{1}{2} \dim V$, we have

$$\dim(U + \sum_{i=1}^{k} T_i(U)) \geq (1 + \varepsilon) \dim U.$$

Dimension expanders – examples

Definition

$(V, T_1, \ldots, T_k)$ is called an ε-expander if, for all $U \subset V$ such that $\dim U \leq \frac{1}{2} \dim V$, we have

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Examples

- No expanders for $k = 1$: take eigenspace U, then $U + T_1(U) = U$.
- For $k \geq 2$ and fixed V, expanders exist for some ε: take T_1, T_2 without common invariant subspaces.

Dimension expanders – examples

Definition

$(V, T_1, \ldots, T_k)$ is called an ε-expander if, for all subspaces $U \subset V$ such that $\dim U \leq \frac{1}{2} \dim V$, we have

$$\dim(U + \sum_{i=1}^{k} T_i(U)) \geq (1 + \varepsilon) \dim U.$$

Examples

- $k = 2$, $\dim V = 4$:
  $U = \langle \text{eigenvector of } T_1 \rangle$: $\dim U + T_1(U) + T_2(U) \leq 2$.
  Using eigenvector of T_1^*: exists two-dimensional U such that $\dim U + T_1(U) + T_2(U) \leq 3$.
  Therefore $\varepsilon = 1/2$ is optimal.

Dimension expanders – results

Dimension expanders are thus a linear algebra analogue of expander graphs. Main result is **uniform** existence:

Theorem (Lubotzky-Zelmanov 2008, Bourgain 2009)

There exist $k \geq 2$ and a **fixed** $\varepsilon > 0$ *such that ε-expanders $(V, T_1, \dots, T_k)$ exist for* **every** *dimension* $\dim V$.

We sharpen this to:

Theorem (R. 2022)

For all $k \geq 2$, there exist ε_k-expanders $(V, T_1, \dots, T_k)$ for every dimension $\dim V$, *where*

$$\varepsilon_k = \frac{1}{2}(k + 1 - \sqrt{k^2 - 2k + 5}),$$

and ϵ_k is optimal.

Q finite quiver, set of vertices Q_0, dimension vectors $\mathbf{d} = (d_i)_i \in \mathbb{N}Q_0$. Euler form

$$\langle \mathbf{d}, \mathbf{e} \rangle_Q = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha: i \to j} d_i e_j.$$

Fix F-vector spaces V_i of dimension d_i for $i \in Q_0$.

$$\prod_{i \in Q_0} \mathrm{GL}(V_i) = G_{\mathbf{d}} \text{ acts on } R_{\mathbf{d}}(Q) = \bigoplus_{\alpha: i \to j} \mathrm{Hom}(V_i, V_j)$$

via base change

$$(g_i)_i \cdot (f_\alpha)_\alpha = (g_j f_\alpha g_i^{-1})_{\alpha: i \to j},$$

such that orbits correspond to isomorphism classes.

Subrepresentations of general representations

For $\mathbf{e} \leq \mathbf{d}$ define $S_{\mathbf{e}} \subset R_{\mathbf{d}}(Q)$ subset of representations admitting subrepresentation of dimension vector $\mathbf{e}$. It is a Zariski-closed subset of $R_{\mathbf{d}}(Q)$.

Definition

Write $\mathbf{e} \hookrightarrow \mathbf{d}$ if $S_{\mathbf{e}} = R_{\mathbf{d}}(Q)$, that is, if every representation of dimension vector $\mathbf{d}$ admits subrepresentation of dimension vector $\mathbf{e}$.

This can be characterized by recursively defined linear inequalities:

Theorem (Schofield, Crawley-Boevey)

We have $\mathbf{e} \hookrightarrow \mathbf{d}$ *iff* $\langle \mathbf{e}', \mathbf{d} - \mathbf{e} \rangle \geq 0$ *for all* $\mathbf{e}' \hookrightarrow \mathbf{e}$.

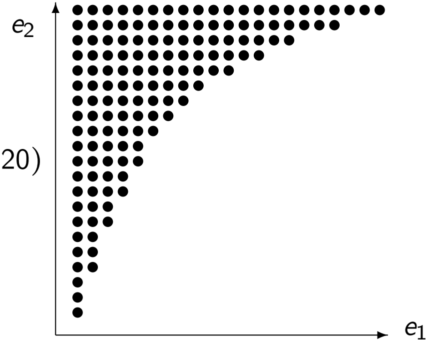
Fix $m \geq 3$. $K(m)$ generalized Kronecker quiver $1 \overset{(m)}{\Rightarrow} 2$.

Theorem

Suppose $\langle \mathbf{d}, \mathbf{d} \rangle \leq 0$. *Then* $\mathbf{e} \hookrightarrow \mathbf{d}$ *iff* $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0$ *iff*

$$e_2 \geq c_{\mathbf{d}}(e_1) := \frac{1}{2}\left(me_1 + d_2 - \sqrt{(me_1 - d_2)^2 + 4e_1(d_1 - e_1)} \right).$$

$m = 4$, $\mathbf{d} = (20, 20)$



Given $(V, T_1, \ldots, T_k \in \mathrm{End}(V))$, define $V(T_*)$ as the representation

$$(\mathrm{id}_V, T_1, \ldots, T_k) : V \overset{(k+1)}{\Rightarrow} V$$

of $K(k+1)$ of dimension vector $\mathbf{d} = (d, d)$.

Observation

There exists an ε-expander on V with k operators iff there is no $\mathbf{e} \hookrightarrow (d, d)$ for $K(k+1)$ such that

$$e_1 \leq \frac{1}{2} d \text{ and } e_2 < (1 + \varepsilon) e_1 \qquad (\star)$$

Proof: In this case, the set $\bigcup_{(\star)} S_{\mathbf{e}} \subset R_{\mathbf{d}}(Q)$ is a finite union of proper Zariski-closed subsets, thus its complement is non-empty (thus open dense in $R_{\mathbf{d}}(Q)$).

Sharp existence result

Using the previous observation and the explicit form of $c_{\mathbf{d}}(x)$:

Theorem

There exist ε-expanders with k operators in **all** *dimensions d iff*

$$\varepsilon \leq \varepsilon_k = \frac{1}{2}(k+1-\sqrt{k^2-2k+5}).$$

Proof: We have to prove

$$c_{\mathbf{d}}(x) \geq (1+\varepsilon)x \text{ for } x \leq \frac{d}{2}.$$

$c_{\mathbf{d}}(x)$ is concave, so suffices to prove for $x = d/2$, RHS equals $\frac{c_{\mathbf{d}}(d/2)}{d/2} - 1$.

Generalization to arbitrary quivers

Now let Q be again an arbitrary (finite, acyclic) quiver.
Choose linear forms $\Theta, \kappa \in \mathbb{R}Q_0^*$ such that κ is positive: $\kappa(\mathbb{R}_{>0}Q_0) \subset \mathbb{R}_{>0}$.
For $0 \neq \mathbf{d} \in \mathbb{N}Q_0$, define $\mu(\mathbf{d}) = \Theta(\mathbf{d})/\kappa(\mathbf{d})$ *slope function*, and $\mu(V) = \mu(\mathbf{d})$ if $V \neq 0$ has dimension vector $\mathbf{d}$.

Definition

- V is μ-stable if $\mu(U) < \mu(V)$ for all $U \subset V$ proper non-zero subrepresentation.
- V is a (δ, ε)-expander representation if $\mu(U) \leq \mu(V) - \varepsilon$ for all $0 \neq U \subset V$ such that $\kappa(U) \leq \delta \cdot \kappa(V)$.

Expander representations

Definition

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- V is a (δ, ε)-expander representation if $\mu(U) \leq \mu(V) - \varepsilon$ for all $0 \neq U \subset V$ such that $\kappa(U) \leq \delta \cdot \kappa(V)$.

Lemma

- *If V is μ-stable, then $\mathrm{End}(V) = F$.*
- *V is μ-stable iff $\forall 0 < \delta < 1\ \exists \varepsilon > 0$: V is a (δ, ε)-expander representation.*

Main theorem

Theorem

If Q is not of (extended) Dynkin type, and only then, there exists a slope function μ for Q and a family $(V^{(k)})_k$ of representations of Q of unbounded dimension such that uniform expansion holds:

$$\forall 0 < \delta < 1\ \exists \varepsilon > 0\text{: every } V^{(k)} \text{ is a } (\delta, \varepsilon)\text{-expander representation.}$$

(Vague) idea of proof: If Q is Dynkin, there are only finitely many indecomposable representations by Gabriel's Theorem. In a uniform expander as in the theorem, every $V^{(k)}$ is μ-stable, thus $\mathrm{End}(V^{(k)}) = F$, thus indecomposable, a contradiction. Similarly for Q extended Dynkin.

In all other cases (Q wild), symmetrized Euler form has negative eigenvalue, and this allows to control the $\mathbf{e} \hookrightarrow \mathbf{d}$ as above (many dirty details missing here).

Final slide

Thank you!