Derived Categories of Coherent Sheaves II

The Bondal–Orlov Reconstruction Theorem

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Introduction

Notation: Let k be a field. All varieties will be k-varieties, typically smooth projective.

Talk 1: Introduction to the **derived category of coherent sheaves**

Notation: For X a variety, write $\mathbf{D^b(\mathit{X})} := \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for the bounded derived category of coherent sheaves on X.

Talk 2 and Talk 3: Discussion of $\mathrm{D}^{\mathrm{b}}(X)$ as an **invariant of X**

More precisely, the guiding question for the remaining talks is the following:

What can we say about X knowing only $\mathrm{D}^{\mathrm{b}}(X)$?

The Bondal–Orlov Reconstruction Theorem

Theorem (Bondal–Orlov)

Let X, Y be smooth projective k-varieties such that there is a k-linear exact equivalence

$$\mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$$

of their derived categories. If $\omega_X^{\pm 1}$ is ample, then $\omega_Y^{\pm 1}$ is ample and $X \cong Y$.

Goals: Give (partial) answers to the following:

- What role does ampleness play in the theorem?
- Why do we consider (ampleness of) the (anti)canonical bundles?
- How do we construct an isomorphism from the equivalence?

Cohomological Reconstruction of Projective Varieties I

The information the derived category of a variety carries is primarily of cohomological nature. Thus, we try to **reconstruct** a variety from its cohomology.

Recall that a **(very) ample line bundle $\mathcal{L}$** on X determines a closed embedding $i\colon X \hookrightarrow \mathbb{P}_k^N$ so that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}_k^N}(1)$. The associated **cohomology ring** is:

$$A_{X,\mathcal{L}} := \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$$

It **reconstructs X** in the sense that $X \cong \operatorname{Proj}\left(A_{X,\mathcal{L}}\right)$.

Cohomological Reconstruction of Projective Space

Example. Let $X = \mathbb{P}^d_k$ and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^d_k}(1)$. Then

$$\mathrm{H}^0(X, \mathcal{L}^{\otimes n}) = \mathrm{H}^0(\mathbb{P}^d_k, \mathcal{O}_{\mathbb{P}^d_k}(n)) \cong k[x_0, \dots, x_d]_n$$

and

$$\mathrm{Proj}\left(\bigoplus\nolimits_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})\right) \cong \mathrm{Proj}\left(\bigoplus\nolimits_{n \geq 0} k[x_0, \dots, x_d]_n\right).$$

In particular, we have $A_{\mathbb{P}^d_k, \mathcal{O}_{\mathbb{P}^d_k}(1)} \cong k[x_0, \dots, x_d]$.

Example. Let $X = \mathbb{P}^d_k$ and let $\mathcal{L} = \omega_X = \mathcal{O}_{\mathbb{P}^d_k}(-d-1)$. Then $\mathcal{L}^{-1} = \omega_X^{-1} = \mathcal{O}_{\mathbb{P}^d_k}(d+1)$ determines a closed embedding $i\colon \mathbb{P}^d_k \hookrightarrow \mathbb{P}^N_k$ for $N = \binom{2n+1}{n+1} - 1$. This is the **Veronese embedding of degree $n+1$**. It realises $\mathbb{P}^d_k$ as a closed subvariety of $\mathbb{P}^N_k$. The **anticanonical ring** $A_{\mathbb{P}^d_k, \omega^{-1}_{\mathbb{P}^d_k}}$ is the associated homogeneous coordinate ring.

Cohomological Reconstruction of Projective Varieties II

Let X, Y be (smooth) projective k-varieties and let

$$F\colon \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$$

be a k-linear exact equivalence.

Suppose that we can reconstruct X from $\mathcal{L}_X$ and we can reconstruct Y from $\mathcal{L}_Y$. A priori, we have no control about what type of complexes $F(\mathcal{L}_X)$ and $F^{-1}(\mathcal{L}_Y)$ are. In fact, there is no reason for $\mathcal{L}_X$ and $\mathcal{L}_Y$ to be related in any way whatsoever.

This is why we consider the **(anti)canonical bundles** $\omega_X^{\pm 1}$ and $\omega_Y^{\pm 1}$. They are intimately tied to the structure of **categories** $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^{\mathrm{b}}(Y)$ via the associated **Serre functors**.

Serre Duality and Serre Functors I

Theorem (Serre Duality)

For any locally free sheaf $\mathcal{E}$ on X and for all i, there are natural isomorphisms of the form:

$$\mathrm{H}^i(X,\mathcal{E}) \cong \mathrm{H}^{\dim X - i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)^\vee$$

Recall that for $\mathcal{F},\mathcal{G} \in \mathsf{Coh}(X)$, the Hom spaces in $\mathrm{D^b}(X)$ encode the Ext functors for $\mathsf{Coh}(X)$: we have $\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{F},\mathcal{G}[i]) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. With this in mind, we can rewrite Serre duality as follows:

$$\begin{aligned}
\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{E}) &\cong \mathrm{H}^i(X,\mathcal{E}) && \Gamma(-) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-) \\
&\cong \mathrm{H}^{\dim X - i}(X,\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)^\vee && \text{Serre duality} \\
&\cong \mathrm{Ext}^{\dim X - i}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega_X)^\vee && \Gamma(-) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-) \\
&\cong \mathrm{Ext}^{\dim X - i}_{\mathcal{O}_X}(\mathcal{E},\omega_X)^\vee && \mathcal{E}\text{ locally free}
\end{aligned}$$

This can now be upgraded to a functor on the entire category $\mathrm{D^b}(X)$.

Serre Duality and Serre Functors II

Definition (Serre Functor)

Let $\mathcal{A}$ be a k-linear category such that $\dim_k \operatorname{Hom}_{\mathcal{A}}(A, B) < \infty$ for all $A, B \in \mathcal{A}$. A **Serre functor on $\mathcal{A}$** is a k-linear equivalence $S \colon \mathcal{A} \to \mathcal{A}$ such that for all $A, B \in \mathcal{A}$ there is an isomorphism

$$\eta_{A,B} \colon \operatorname{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(B, S(A))^{\vee}$$

natural in A and B.

Compare with the last slide:

$$\operatorname{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X, \mathcal{E}[i]) \cong \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}) \overset{\substack{\text{Serre} \\ \text{duality}}}{\cong} \operatorname{Ext}^{\dim X - i}_{\mathcal{O}_X}(\mathcal{E}, \omega_X)^{\vee} \cong \operatorname{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}, \omega_X[\dim X - i])^{\vee}$$

$S_X \colon \mathcal{E}^{\bullet} \mapsto (\mathcal{E}^{\bullet} \otimes_{\mathcal{O}_X} \omega_X)[\dim X]$ defines a Serre functor on $\mathrm{D^b}(X)$

Key remark: If $F \colon \mathcal{A} \to \mathcal{B}$ is **any** k-linear equivalence and $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ are Serre functors on $\mathcal{A}$ and $\mathcal{B}$, respectively, then $S_{\mathcal{B}} \circ F \cong F \circ S_{\mathcal{A}}$.

Proof Sketch

Let $F\colon \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$ be a k-linear exact equivalence. It can be shown that $\dim X = \dim Y$.
If $F(\mathcal{O}_X) \cong \mathcal{O}_Y$, **then** $F(\omega_X) \cong \omega_Y$:

$$F(\omega_X) = F(S_X(\mathcal{O}_X)[-\dim X]) \overset{\substack{\text{Key}\\\text{Remark}}}{\cong} S_Y(F(\mathcal{O}_X))[-\dim Y] = S_Y(\mathcal{O}_Y)[-\dim Y] = \omega_Y$$

In fact, $F(\omega_X^{\pm n}) = \omega_Y^{\pm n}$ for all $n \in \mathbb{N}$. Thus, the **equivalence** F induces an **isomorphism**

$$\bigoplus_{n\geq 0} \mathrm{H}^0(X, \omega_X^{\pm n}) \cong \bigoplus_{n\geq 0} \mathrm{H}^0(Y, \omega_Y^{\pm n})$$

of the associated (anti)canonical rings. **If** we can show that $\omega_X^{\pm 1}$ being ample implies that $\omega_Y^{\pm 1}$ is ample, **then** we are done.

The **If**'s of the highlighted **If**..., **then**... phrases are the remaining technical heart of the proof. If time permits, we will sketch some further ideas going into their proofs at the end.

Example: Curves

Let us consider the case where $X = C$ is a smooth projective curve as an example. The deciding invariant is its **genus** $\boldsymbol{g}$, which may be defined as $g = \dim_k \mathrm{H}^1(C, \mathcal{O}_C) = \dim_k \mathrm{H}^0(C, \omega_C)$.

$\boldsymbol{g = 0}$**:** If $g = 0$, then $C \cong \mathbb{P}^1_k$. In particular, the anticanonical bundle is $\omega_C^{-1} \cong \mathcal{O}_C(2)$, hence ample. Thus, if $\mathrm{D^b}(C) \simeq \mathrm{D^b}(X)$ for some smooth projective variety X, then $C \cong X$ by the **reconstruction theorem**.

$\boldsymbol{g = 1}$**:** If $g = 1$, then $C \cong E$ is an elliptic curve, hence an abelian variety. In particular, $\omega_C \cong \mathcal{O}_C$ is trivial, hence not ample, and the **reconstruction theorem** does **NOT** apply. Nonetheless, it remains true that if $\mathrm{D^b}(E) \simeq \mathrm{D^b}(X)$ for some smooth projective variety X, then $E \cong X$. This uses the machinery of **Fourier–Mukai transforms**.

$\boldsymbol{g > 1}$**:** If $g > 1$, then $\omega_C \cong \mathcal{O}_C(2g-2)$ and $2g - 2 > 0$. In particular, the canonical bundle ω_C is ample. Thus, if $\mathrm{D^b}(C) \simeq \mathrm{D^b}(X)$ for some smooth projective variety X, then $C \cong X$ by the **reconstruction theorem**.

More Details for the Proof Sketch I

One key step is to give a **cohomological interpretation** of the points of a smooth projective variety and of its line bundles.

An object $\mathcal{P}^\bullet \in \mathrm{D}^\mathrm{b}(X)$ is **point-like (of codimension d)**, if $S_X(\mathcal{P}^\bullet) \cong \mathcal{P}^\bullet[d]$ and

$$\mathrm{Hom}_{\mathrm{D}^\mathrm{b}(X)}(\mathcal{P}^\bullet, \mathcal{P}^\bullet[i]) = \begin{cases} \text{a field denoted } \kappa(\mathcal{P}^\bullet) & i = 0 \\ 0 & i < 0 \end{cases}$$

An object $\mathcal{L}^\bullet \in \mathrm{D}^\mathrm{b}(X)$ is **invertible** if for each point-like object $\mathcal{P}^\bullet \in \mathrm{D}^\mathrm{b}(X)$ there is an integer $n_{\mathcal{P}^\bullet}$ such that

$$\mathrm{Hom}_{\mathrm{D}^\mathrm{b}(X)}(\mathcal{L}^\bullet, \mathcal{P}^\bullet[i]) = \begin{cases} \kappa(\mathcal{P}^\bullet) & i = n_{\mathcal{P}^\bullet} \\ 0 & i \neq n_{\mathcal{P}^\bullet} \end{cases}$$

One then shows that objects of the form $\kappa(x)[m]$, with $x \in X$ a closed point and $m \in \mathbb{Z}$, are point-like and that any invertible object is of the form $\mathcal{L}[m]$, with $\mathcal{L} \in \operatorname{Pic} X$ and $m \in \mathbb{Z}$. **If** $\omega_X^{\pm 1}$ is ample, **then** the converse hold in each case (see next slide).

More Details for the Proof Sketch II

Let $F\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ be as before. We obtain the following:

$$\begin{tikzcd}
\{\text{point-like objects in } \mathrm{D^b}(X)\} \arrow[r, leftrightarrow, "1:1"] \arrow[d, equal] & \{\text{point-like objects in } \mathrm{D^b}(Y)\} \\
\{\kappa(x)[m] \mid x \in X,\ m \in \mathbb{Z}\} & \{\kappa(y)[m] \mid y \in Y,\ m \in \mathbb{Z}\} \arrow[u, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
\{\text{invertible objects in } \mathrm{D^b}(X)\} \arrow[r, leftrightarrow, "1:1"] \arrow[d, equal] & \{\text{invertible objects in } \mathrm{D^b}(Y)\} \arrow[d, hook] \\
\{\mathcal{L}[m] \mid \mathcal{L} \in \operatorname{Pic} X,\ m \in \mathbb{Z}\} & \{\mathcal{L}[m] \mid \mathcal{L} \in \operatorname{Pic} Y,\ m \in \mathbb{Z}\}
\end{tikzcd}$$

Note that $F(\mathcal{O}_X)$ is an invertible object of $\mathrm{D^b}(Y)$, hence of the form $\mathcal{L}[m]$. Twisting F by $\mathcal{L}[m]$ we may assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. To conclude, it remains to show that **if** $\omega_X^{\pm 1}$ ample, **then** $\omega_Y^{\pm 1}$ is ample. This is quite technical and goes beyond the scope of this talk.

Thanks!