

Derived Categories of Coherent Sheaves I

Short intro to derived categories

Jan Hennig, Julian Reichardt, Fabian Rodatz



**BERGISCHE
UNIVERSITÄT
WUPPERTAL**



16.09.2026

Construction

X an object of interest (e.g. a scheme)

$\mathcal{A}$ an abelian category (e.g. (quasi)coherent sheaves on X)

$Ch(\mathcal{A})$ category of chain complexes

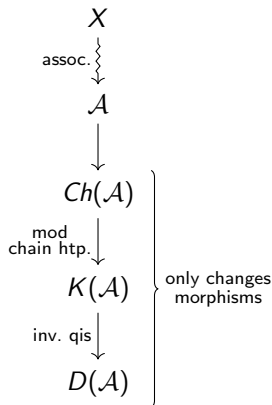
$K(\mathcal{A})$ homotopy category: $Hom_{K(\mathcal{A})}(C, D) = Hom_{Ch(\mathcal{A})}(C, D) / \simeq$
 $f \simeq g$ chain homotopic: $f^i - g^i = d_D^{i-1} \circ s_i + s_{i+1} \circ d_C^i$

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & C^{i-1} & \longrightarrow & C^i & \longrightarrow & C^{i+1} \longrightarrow \dots \\
 & & \Downarrow & & \downarrow f^i & \downarrow g^i & \Downarrow \\
 \dots & \longrightarrow & D^{i-1} & \longrightarrow & D^i & \longrightarrow & D^{i+1} \longrightarrow \dots
 \end{array}$$

$\swarrow \text{dashed } s_i \quad \searrow \text{dashed } s_{i+1}$

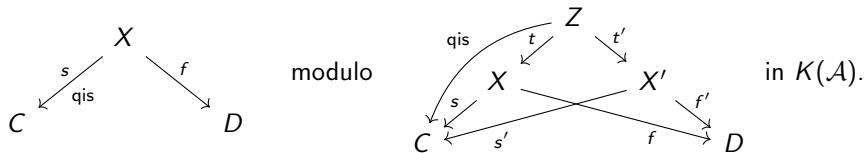
$D(\mathcal{A}) = [qis]^{-1}K(\mathcal{A})$ the derived category of $\mathcal{A}$

$f: C \rightarrow D$ quasi-isomorphism: $H^i(f): H^i(C) \xrightarrow{\cong} H^i(D)$ for all i



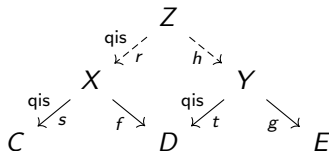
Morphisms

A morphism $C \rightarrow D$ in $D(\mathcal{A})$ is given by a “roof” $f \circ s^{-1}$:



Calculus of fractions: $f \circ s^{-1} = f' \circ (s')^{-1}$ means
there is a common denominator $s \circ t = s' \circ t'$ (qis) such that the nominators $f \circ t = f' \circ t'$ agree.

Composition:



$$(g \circ t^{-1}) \circ (f \circ s^{-1}) = (g \circ h) \circ (s \circ r)^{-1}$$

Bounded derived categories

There are bounded versions:

$$D^+(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \text{ for } i < m\}$$

$$D^-(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \text{ for } i > m\}$$

$$D^b(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \text{ for } |i| > m\}$$

These can be seen as full subcategories in $D(\mathcal{A})$ or or as the appropriate $[qis]^{-1}K^{+,-,b}(\mathcal{A})$.

Resolutions

An object I in $\mathcal{A}$ is called injective if $\begin{array}{ccc} T & \xrightarrow{\forall} & S \\ \forall \downarrow & \swarrow \exists & \\ I & & \end{array}$ (is equivalent to: $\text{Hom}_{\mathcal{A}}(-, I)$ is exact)

An injective resolution of an object $A \in \mathcal{A}$ is an exact sequence

$$\dots \rightarrow 0 \rightarrow A \xrightarrow{\phi} I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots \quad \text{with } I^j \text{ injective.}$$

This gives a quasi-isomorphism $A \rightarrow I^*$:

$$\begin{array}{ccccccc} \dots & \rightarrow & 0 & \rightarrow & A & \rightarrow & 0 \rightarrow 0 \rightarrow \dots \\ & & \downarrow & & \phi \downarrow & & \downarrow \quad \downarrow \\ \dots & \rightarrow & 0 & \rightarrow & I^0 & \rightarrow & I^1 \rightarrow I^2 \rightarrow \dots \end{array}$$

Suppose $\mathcal{A}$ has enough injectives. Then for any $C \in K^+(\mathcal{A})$ there exists a complex $I^* \in K^+(\mathcal{A})$ with $I^j \in \mathcal{A}$ injective and a quasi-isomorphism $C \rightarrow I^*$.

Moreover, $K^+(\mathcal{I}) \rightarrow D^+(\mathcal{A})$ is an equivalence for $\mathcal{I} \subseteq \mathcal{A}$ the full additive subcategory of all injectives of $\mathcal{A}$ (needs: $\text{Hom}_{K^+(\mathcal{A})}(A, I) = \text{Hom}_{D^+(\mathcal{A})}(A, I)$ for $I \in \mathcal{I}$).

Derived functors

Suppose $\mathcal{A}$ has enough injectives and $F: \mathcal{A} \rightarrow \mathcal{B}$ is a left-exact functor
(i.e. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact implies $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$ exact)

We get an (exact) functor $RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ the right-derived functor in the following way

$$RF = Q_B \circ K(F) \circ \iota^{-1}: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B}),$$

$$\begin{array}{ccccc}
 K^+(\mathcal{I}_{\mathcal{A}}) & \hookrightarrow & K^+(\mathcal{A}) & \xrightarrow{K(F)} & K^+(\mathcal{B}) \\
 & \searrow \iota & \downarrow Q_A & & \downarrow Q_B \\
 & & D^+(\mathcal{A}) & & D^+(\mathcal{B})
 \end{array}$$

(Note: A curved arrow labeled ι^{-1} points from $D^+(\mathcal{A})$ back to $K^+(\mathcal{I}_{\mathcal{A}})$)

i.e. pick an injective resolution, apply F term-wise, look at the complex in $D^+(\mathcal{B})$.

Get the higher derived functors $R^i F: \mathcal{A} \rightarrow \mathcal{B}$, $A \mapsto H^i(RF(A))$

For an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ get a long exact sequence

$$0 \rightarrow \underbrace{R^0 F(A)}_{=F(A)} \rightarrow \underbrace{R^0 F(B)}_{=F(B)} \rightarrow \underbrace{R^0 F(C)}_{=F(C)} \rightarrow R^1 F(A) \rightarrow R^1 F(B) \rightarrow R^1 F(C) \rightarrow R^2 F(A) \rightarrow \dots$$

Dual story

The dual story works too...

Injective become projective, $K^+(\mathcal{A})$ becomes $K^-(\mathcal{A})$, ...

Suppose $\mathcal{A}$ has enough projectives and $F: \mathcal{A} \rightarrow \mathcal{B}$ is a right-exact functor
(i.e. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact implies $F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$ exact)

We get an (exact) functor $LF: D^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$ the left-derived functor in the following way

$$LF = Q_B \circ K(F) \circ \iota^{-1}: D^-(\mathcal{A}) \rightarrow D^-(\mathcal{B}),$$

i.e. pick a projective resolution, apply F term-wise, look at the complex in $D^-(\mathcal{B})$.

Get the higher derived functors $L^i F: \mathcal{A} \rightarrow \mathcal{B}$, $A \mapsto H^i(LF(A))$

For an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ get a long exact sequence

$$\cdots \rightarrow L^2 F(C) \rightarrow L^1 F(A) \rightarrow L^1 F(B) \rightarrow L^1 F(C) \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

Example

Let $A, B \in \mathcal{A}$. What is $\text{Hom}_{D(\mathcal{A})}(A, B[i])$?

$$\begin{array}{cccccccccccccccc}
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \swarrow & \downarrow & & \downarrow & \\
 \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^{i-1} & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow & \dots
 \end{array}$$

$$\text{Hom}_{D(\mathcal{A})}(A, B[i]) = \text{Hom}_{D(\mathcal{A})}(A, I^*[i]) = \text{Hom}_{K(\mathcal{A})}(A, I^*[i]) = \text{Hom}_{Ch(\mathcal{A})}(A, I^*[i]) / \simeq$$

$$\begin{aligned}
 f \in \text{Hom}_{\mathcal{A}}(A, I^i) \text{ morphism of chain complexes} &\iff f \in \ker(\text{Hom}_{\mathcal{A}}(A, I^i) \rightarrow \text{Hom}_{\mathcal{A}}(A, I^{i+1})) \\
 f \simeq 0 &\iff f \in \text{im}(\text{Hom}_{\mathcal{A}}(A, I^{i-1}) \rightarrow \text{Hom}_{\mathcal{A}}(A, I^i))
 \end{aligned}$$

Therefore: $\text{Hom}_{D(\mathcal{A})}(A, B[i]) = \text{Ext}^i(A, B) = R^i \text{Hom}(A, -)(B).$

Derived category of coherent sheaves

Let X be a scheme (noetherian/smooth/proper/projective/finite type over k).

$D^b(X) := D^b(\text{Coh}(X))$ the bounded derived category of coherent sheaves on X

Problem: $\text{Coh}(X)$ does not admit enough projective/injective resolutions

(Solution: work in $\text{QCoh}(X)$ or use different resolutions (e.g. flabby, locally free, ...))

$$\begin{array}{lll} f: X \rightarrow Y \text{ proper} & \rightsquigarrow & Rf_*: D^b(X) \rightarrow D^b(Y) \\ X \text{ smooth} & \rightsquigarrow & R\underline{\text{Hom}}(-, -): D^b(X)^{op} \times D^b(X) \rightarrow D^b(X) \\ X \text{ smooth} & \rightsquigarrow & - \otimes^L -: D^b(X) \times D^b(X) \rightarrow D^b(X) \\ & & Lf^*: D^b(Y) \rightarrow D^b(X) \end{array}$$

Compatibilities

Let X, Y be smooth projective schemes over k .

Projection formula: For $f: X \rightarrow Y$ proper

$$Rf_*(\mathcal{F}) \otimes^L \mathcal{E} \xrightarrow{\sim} Rf_*(\mathcal{F} \otimes^L Lf^*(\mathcal{E}))$$

Flat base change: For pullback square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{v} & Y \\ g \downarrow & & \downarrow f \text{ proper} \\ X & \xrightarrow{u \text{ flat}} & Z \end{array}$$

$$u^* Rf_*(\mathcal{F}) \xrightarrow{\sim} Rg_* v^*(\mathcal{F})$$