Derived Categories of Coherent Sheaves I

Short intro to derived categories

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Construction

$$\begin{tikzcd} X \arrow[d, squiggly, "\text{assoc.}"'] \\ \mathcal{A} \arrow[d] \\ Ch(\mathcal{A}) \arrow[d, "\text{mod chain htp.}"'] \\ K(\mathcal{A}) \arrow[d, "\text{inv. qis}"'] \\ D(\mathcal{A}) \end{tikzcd}$$

$\left.\begin{matrix} Ch(\mathcal{A}) \\ K(\mathcal{A}) \\ D(\mathcal{A}) \end{matrix}\right\}$ only changes morphisms

X an object of interest (e.g. a scheme)

$\mathcal{A}$ an abelian category (e.g. (quasi)coherent sheaves on X)

$Ch(\mathcal{A})$ category of chain complexes

$K(\mathcal{A})$ homotopy category: $Hom_{K(\mathcal{A})}(C, D) = Hom_{Ch(\mathcal{A})}(C, D)/_{\simeq}$
$f \simeq g$ chain homotopic: $f^i - g^i = d_D^{i-1} \circ s_i + s_{i+1} \circ d_C^i$

$$\begin{tikzcd} \dots \arrow[r] & C^{i-1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & C^i \arrow[r] \arrow[d, shift right, "f^i"'] \arrow[d, shift left, "g^i"] \arrow[dl, dashed, "s_i"'] & C^{i+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, dashed, "s_{i+1}"] & \dots \\ \dots \arrow[r] & D^{i-1} \arrow[r] & D^i \arrow[r] & D^{i+1} \arrow[r] & \dots \end{tikzcd}$$

$D(\mathcal{A}) = [qis]^{-1} K(\mathcal{A})$ the derived category of $\mathcal{A}$
$f : C \to D$ quasi-isomorphism: $H^i(f) : H^i(C) \xrightarrow{\cong} H^i(D)$ for all i

Morphisms

A morphism $C \to D$ in $D(\mathcal{A})$ is given by a "roof" $f \circ s^{-1}$:

$$\begin{tikzcd} & X \arrow[dl, "s"', "\text{qis}"] \arrow[dr, "f"] & \\ C & & D \end{tikzcd} \quad \text{modulo} \quad \begin{tikzcd} & & Z \arrow[dl, "t"'] \arrow[dr, "t'"] \arrow[ddll, bend right, "\text{qis}"'] & & \\ & X \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X' \arrow[dr, "f'"] \arrow[dlll, "s'" near end] & \\ C & & & & D \end{tikzcd} \quad \text{in } K(\mathcal{A}).$$

Calculus of fractions: $f \circ s^{-1} = f' \circ (s')^{-1}$ means
there is a common denominator $s \circ t = s' \circ t'$ (qis) such that the nominators $f \circ t = f' \circ t'$ agree.

Composition:

$$\begin{tikzcd} & & Z \arrow[dl, dashed, "\text{qis}"', "r"] \arrow[dr, dashed, "h"'] & & \\ & X \arrow[dl, "\text{qis}"', "s"] \arrow[dr, "f"'] & & Y \arrow[dl, "\text{qis}"', "t"] \arrow[dr, "g"'] & \\ C & & D & & E \end{tikzcd}$$

$$(g \circ t^{-1}) \circ (f \circ s^{-1}) = (g \circ h) \circ (s \circ r)^{-1}$$

Bounded derived categories

There are bounded versions:

$$D^+(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \quad \text{for } i < m\}$$
$$D^-(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \quad \text{for } i > m\}$$
$$D^b(\mathcal{A}) = \{C \in D(\mathcal{A}) \mid \exists m : H^i(C) = 0 \quad \text{for } |i| > m\}$$

These can be seen as full subcategories in $D(\mathcal{A})$ or or as the appropriate $[qis]^{-1}K^{+,-,b}(\mathcal{A})$.

Resolutions

An object I in $\mathcal{A}$ is called injective if
$$\begin{tikzcd} T \arrow[r, hook, "\forall"] \arrow[d, "\forall"'] & S \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd}$$
(is equivalent to: $Hom_{\mathcal{A}}(-, I)$ is exact)

An injective resolution of an object $A \in \mathcal{A}$ is an exact sequence

$$\ldots \longrightarrow 0 \longrightarrow A \xrightarrow{\phi} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \ldots \qquad \text{with } I^j \text{ injective.}$$

This gives a quasi-isomorphism $A \to I^*$:
$$\begin{array}{ccccccccccc} \ldots & \to & 0 & \to & A & \to & 0 & \to & 0 & \to & \ldots \\ & & \downarrow & & \phi\downarrow & & \downarrow & & \downarrow & & \\ \ldots & \to & 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to & \ldots \end{array}$$

Suppose $\mathcal{A}$ has enough injectives. Then for any $C \in K^+(\mathcal{A})$ there exists a complex $I^* \in K^+(\mathcal{A})$ with $I^j \in \mathcal{A}$ injective and a quasi-isomorphism $C \to I^*$.

Moreover, $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence for $\mathcal{I} \subseteq \mathcal{A}$ the full additive subcategory of all injectives of $\mathcal{A}$ (needs: $Hom_{K^+(\mathcal{A})}(A, I) = Hom_{D^+(\mathcal{A})}(A, I)$ for $I \in \mathcal{I}$).

Derived functors

Suppose $\mathcal{A}$ has enough injectives and $F\colon \mathcal{A} \to \mathcal{B}$ is a left-exact functor
(i.e. $0 \to A \to B \to C \to 0$ exact implies $0 \to F(A) \to F(B) \to F(C)$ exact)

We get an (exact) functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ the right-derived functor in the following way

$$RF = Q_B \circ K(F) \circ \iota^{-1}\colon D^+(\mathcal{A}) \to D^+(\mathcal{B}),$$

$$\begin{tikzcd}
K^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] \arrow[dr, "\iota"] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_A"] & K^+(\mathcal{B}) \arrow[d, "Q_B"] \\
 & D^+(\mathcal{A}) \arrow[ul, bend left, "\iota^{-1}"] & D^+(\mathcal{B})
\end{tikzcd}$$

i.e. pick an injective resolution, apply F term-wise, look at the complex in $D^+(\mathcal{B})$.

Get the higher derived functors $R^iF\colon \mathcal{A} \to \mathcal{B}, \quad A \mapsto H^i(RF(A))$
For an exact sequence $0 \to A \to B \to C \to 0$ get a long exact sequence

$$0 \to \underbrace{R^0F(A)}_{=F(A)} \to \underbrace{R^0F(B)}_{=F(B)} \to \underbrace{R^0F(C)}_{=F(C)} \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

Dual story

The dual story works too...

Injective become projective, $K^+(\mathcal{A})$ becomes $K^-(\mathcal{A})$, ...

Suppose $\mathcal{A}$ has enough projectives and $F\colon \mathcal{A} \to \mathcal{B}$ is a right-exact functor
(i.e. $0 \to A \to B \to C \to 0$ exact implies $F(A) \to F(B) \to F(C) \to 0$ exact)

We get an (exact) functor $LF\colon D^-(\mathcal{A}) \to D^-(\mathcal{B})$ the left-derived functor in the following way

$$LF = Q_B \circ K(F) \circ \iota^{-1}\colon D^-(\mathcal{A}) \to D^-(\mathcal{B}),$$

i.e. pick a projective resolution, apply F term-wise, look at the complex in $D^-(\mathcal{B})$.

Get the higher derived functors $L^iF\colon \mathcal{A} \to \mathcal{B}, \quad A \mapsto H^i(LF(A))$
For an exact sequence $0 \to A \to B \to C \to 0$ get a long exact sequence

$$\cdots \to L^2F(C) \to L^1F(A) \to L^1F(B) \to L^1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

Example

Let $A, B \in \mathcal{A}$. What is $Hom_{D(\mathcal{A})}(A, B[i])$?

$$
\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \dots \arrow[r] & I^{i-1} \arrow[r] & I^i \arrow[r] & I^{i+1} \arrow[r] & \dots
\end{tikzcd}
$$

$$Hom_{D(\mathcal{A})}(A, B[i]) = Hom_{D(\mathcal{A})}(A, I^*[i]) = Hom_{K(\mathcal{A})}(A, I^*[i]) = Hom_{Ch(\mathcal{A})}(A, I^*[i])/ \simeq$$

$$f \in Hom_{\mathcal{A}}(A, I^i) \text{ morphism of chain complexes } \iff f \in \ker\left(Hom_{\mathcal{A}}(A, I^i) \to Hom_{\mathcal{A}}(A, I^{i+1})\right)$$

$$f \simeq 0 \iff f \in im(Hom_{\mathcal{A}}(A, I^{i-1}) \to Hom_{\mathcal{A}}(A, I^i))$$

Therefore: $Hom_{D(\mathcal{A})}(A, B[i]) = Ext^i(A, B) = R^i Hom(A, -)(B).$

Derived category of coherent sheaves

Let X be a scheme (noetherian/smooth/proper/projective/finite type over k).

$$D^b(X) := D^b(Coh(X)) \qquad \text{the bounded derived category of coherent sheaves on } X$$

Problem: $Coh(X)$ does not admit enough projective/injective resolutions
(Solution: work in $QCoh(X)$ or use different resolutions (e.g. flabby, locally free, ...))

$$\begin{aligned}
f\colon X \to Y \text{ proper} \quad &\rightsquigarrow \quad Rf_*\colon D^b(X) \to D^b(Y)\\
X \text{ smooth} \quad &\rightsquigarrow \quad R\underline{Hom}(-,-)\colon D^b(X)^{op} \times D^b(X) \to D^b(X)\\
X \text{ smooth} \quad &\rightsquigarrow \quad - \otimes^L -\colon D^b(X) \times D^b(X) \to D^b(X)\\
& \qquad\quad Lf^*\colon D^b(Y) \to D^b(X)
\end{aligned}$$

Compatibilities

Let X, Y be smooth projective schemes over k.

Projection formula: For $f : X \to Y$ proper

$$Rf_*(\mathcal{F}) \otimes^L \mathcal{E} \xrightarrow{\sim} Rf_*(\mathcal{F} \otimes^L Lf^*(\mathcal{E}))$$

Flat base change: For pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"'] & Y \arrow[d, "f \text{ proper}"] \\ X \arrow[r, "u \text{ flat}"'] & Z \end{tikzcd}$$

$$u^* Rf_*(\mathcal{F}) \xrightarrow{\sim} Rg_* v^*(\mathcal{F})$$