Derived Categories of Coherent Sheaves III

Fourier-Mukai Transforms of Abelian Varieties

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Introduction

Disclaimer: k algebraically closed field of characteristic 0, e.g. $k = \mathbb{C}$.
For a (smooth projective) variety X we can associate the bounded derived category of coherent sheaves

$$X \rightsquigarrow \mathrm{D}^b(X)$$

Theorem: (Bondal-Orlov 2001) Let X, Y smooth projective varieties & with ample (anti) canonical bundle $\omega_X^{\pm 1}$. If $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$, then $X \cong Y$.

Question

Are there non-isomorphic smooth projective varieties $X \ncong Y$ such that $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$?

Answer: Yes! There are abelian varieties A with $A \ncong \hat{A}$, but $\mathrm{D}^b(A) \simeq \mathrm{D}^b(\hat{A})$ (Mukai 1981).

Goal: 1) Introduce abelian varieties $A, \hat{A}$ and useful tools
2) Understand and proof the equivalence $\mathrm{D}^b(A) \simeq \mathrm{D}^b(\hat{A})$

Abelian Variety

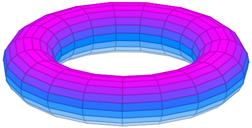
Definition

An **abelian variety** is a (smooth) projective[a] variety A that is also an (abelian) connected algebraic group (i.e. $m\colon A \times_k A \to A, \iota\colon A \to A$ and $e\colon \mathrm{Spec}(k) \to A$ such that group axioms).

[a]or complete

Remark:

- An abelian variety of dimension $d = 1$ is an **elliptic curve**
- For $k = \mathbb{C}$ the associated complex manifold $A(\mathbb{C})$ is a torus $\mathbb{C}^d/\Lambda$
  e.g. $\mathbb{C}/\mathbb{Z}^2$



- $\omega_A = \mathcal{O}_A$ is trivial (i.e. very non-ample)

Dual abelian variety and Poincaré Bundle

- **Definition:**
  - $\mathrm{Pic}^0(A) := \{[L] \in \mathrm{Pic}(A) \mid T_a^* L \cong L \quad \forall a \in A(k)\}$ with $T_a \colon A \to A, b \mapsto m(a,b)$ translation
  - define the functor

$$\mathcal{F}_A \colon \mathrm{Schemes}/k \to \mathrm{Sets}$$
$$S \mapsto \{[L] \in \mathrm{Pic}(A \times S) \mid L|_{\{0\}\times S} = \mathcal{O}_S, L|_{A\times\{s\}} \in \mathrm{Pic}^0(A) \quad \forall s \in S(k)\}$$

- **Theorem:** The functor $\mathcal{F}_A$ is represented by
  - an abelian variety $\hat{A}$ called the **dual abelian variety**
  - with **universal object** $P \in \mathcal{F}_A(\hat{A})$ called the **Poincaré bundle**.
- **Remark:**
  - $\hat{A}(k) = \mathrm{Pic}^0(A)$ and $P \in \mathrm{Pic}(A \times \hat{A})$ with $P|_{\{0\}\times\hat{A}} = \mathcal{O}_{\hat{A}}$ and $P|_{A\times\{L\}} = L$ for all $L \in \hat{A}(k)$,
  - $A \cong \hat{\hat{A}}$ such that $\hat{P} \in \mathrm{Pic}(\hat{A} \times A)$ is isomorphic to $P \in \mathrm{Pic}(A \times \hat{A})$ under $A \times \hat{A} \xrightarrow{\cong} \hat{A} \times A$.
- **Example:** For E elliptic curve $\hat{E} \cong E$, but for many abelian varieties $\hat{A} \ncong A$.
- **Theorem:** $R\pi_{1*}P = \kappa(0)[-d] \in \mathrm{D}^b(A)$ for $\pi_1 \colon A \times \hat{A} \to A$

Seesaw Principle: Line bundles on product spaces $A \times T$

Theorem (Seesaw Principle)

Let A, T varieties, A complete and $L \in \mathrm{Pic}(L \times T)$. Let $A \times T \xrightarrow{\pi_2} T$.

- *If $L|_{A \times \{t\}}$ trivial for all $t \in T(k)$, then $\exists N \in \mathrm{Pic}(T)$ such that $L \cong \pi_2^* N$;*
- *if furthermore, there is $a \in A(k)$ such that $L|_{\{a\} \times T}$ trivial, then L trivial.*

Corollary: For $L \in \mathrm{Pic}(A)$ we have in $\mathrm{Pic}(A \times A)$

$$m^* L = \pi_1^* L \otimes \pi_2^* L \Leftrightarrow L \in \mathrm{Pic}^0(A).$$

Proof:

- Restricting $m^* L = \pi_1^* L \otimes \pi_2^* L$ to $A \times \{a\}$ gives $T_a^* L = L$ for all $a \in A(k)$;
- Restricting $m^* L = \pi_1^* L \otimes \pi_2^* L$ to $\{e\} \times A$ gives $L = L$.
- by the Seesaw Principle we are done.

Fourier-Mukai Transforms (Mukai 1981)

Let X, Y be smooth projective varieties and let $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ the projections.

- **Definition:** Define the **Fourier-Mukai Transform** with kernel $P \in \mathrm{D}^b(X \times Y)$ as the functor
$$\Phi_P^{X \to Y} : \mathrm{D}^b(X) \xrightarrow{\pi_1^*} \mathrm{D}^b(X \times Y) \xrightarrow{- \otimes P} \mathrm{D}^b(X \times Y) \xrightarrow{\pi_{2*}} \mathrm{D}^b(Y).$$
(all functors are derived)
- **Example:**
  - $- \otimes L[n] = \Phi^{X \to X}_{\mathrm{diag}_* L[n]}$ for $L \in \mathrm{Pic}(X)$ and $n \in \mathbb{Z}$, $\mathrm{diag} : A \to A \times A$ diagonal map;
  - $f_* = \Phi^{X \to Y}_{\mathcal{O}_{\Gamma_f}}$ for $f : X \to Y$ and $\mathcal{O}_{\Gamma_f} := (\mathrm{id}_X \times f)_* \mathcal{O}_X$.
  - $f^* = \Phi^{Y \to X}_{\mathcal{O}_{\Gamma_f^t}}$ for $f : X \to Y$ and $\mathcal{O}_{\Gamma_f^t} := (f \times \mathrm{id}_Y)_* \mathcal{O}_X$.
- **Proof:**
$$\begin{aligned}
\Phi^{Y \to X}_{\mathcal{O}_{\Gamma_f^t}}(-) &= \pi_{2*}\left(\pi_1^*(-) \otimes (f \times \mathrm{id}_Y)_* \mathcal{O}_X\right) \\
&\overset{PF}{=} \pi_{2*}(f \times \mathrm{id}_Y)_* \left((f \times \mathrm{id}_Y)^* \pi_1^*(-) \otimes \mathcal{O}_X\right) \\
&= (\pi_2 \circ (f \times \mathrm{id}_Y))_* \left((\pi_1 \circ (f \times \mathrm{id}_Y))^*(-)\right) \\
&= \mathrm{id}_{Y*} \circ f^*(-) \\
&= f^*(-)
\end{aligned}$$

Composition of FMT

Proposition

The composition

$$\mathrm{D}^b(X) \xrightarrow{\Phi_P} \mathrm{D}^b(Y) \xrightarrow{\Phi_Q} \mathrm{D}^b(Z)$$

is given by the Fourier-Mukai Transform Φ_{Q*P} with kernel

$$Q * P := \pi_{13*}(\pi_{12}^* P \otimes \pi_{23}^* Q)$$

Proof Ingredients: Projection formula, Flat Base Change.

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"'] \arrow[dr, "\pi_{23}"] & \\
X \times Y & X \times Z & Y \times Z
\end{tikzcd}$$

Main Result (Mukai 1981)

A abelian variety; $\hat{A}$ dual; $P \in \mathrm{Pic}(A \times \hat{A})$, $\hat{P} \in \mathrm{Pic}(\hat{A} \times A)$ Poincaré bundles. Define the FMT's

$$\Phi_P^{A \to \hat{A}} \colon \mathrm{D}^b(A) \to \mathrm{D}^b(\hat{A}), \quad \Phi_{\hat{P}}^{\hat{A} \to A} \colon \mathrm{D}^b(\hat{A}) \to \mathrm{D}^b(A)$$

with kernel P and $\hat{P}$

Theorem (Mukai 1981)

$$\Phi_{\hat{P}}^{\hat{A} \to A} \circ \Phi_P^{A \to \hat{A}} = \iota^*[-d].$$

for $\iota \colon A \to A$ the inversion map and $d := \dim(A)$.

Corollary (Mukai 1981)

$\Phi_P^{A \to \hat{A}} \colon \mathrm{D}^b(A) \xrightarrow{\simeq} \mathrm{D}^b(\hat{A})$ *is an equivalence of categories.*

But for many abelian varieties $A \ncong \hat{A}$...!

The Proof of $\Phi^{\hat{A}\to A}_{\hat{P}} \circ \Phi^{A\to\hat{A}}_{P} = \iota^*[-d]$

The composition of the two FMT $\Phi^{\hat{A}\to A}_{\hat{P}} \circ \Phi^{A\to\hat{A}}_{P}$ is given by $\Phi^{A\to A}_{\hat{P}*P}$ with

$$\hat{P} * P = \pi_{13*}(\pi_{12}^* P \otimes \pi_{23}^* \hat{P}) = \pi_{13*}(\tilde{m}^* P) = m^*(\pi_{1*} P)$$
$$= m^*(\kappa(0)[-d]) = (\iota \times \mathrm{id})_*(\mathcal{O}_A)[-d] =: \mathcal{O}_{\Gamma_\iota^t}[-d]$$

As by the Seesaw Principle for $\tilde{m}\colon A \times \hat{A} \times A \to A \times \hat{A}$

$$\pi_{12}^* P \otimes \pi_{23}^* \hat{P} = \tilde{m}^* P \Leftrightarrow \pi_1^*(P|_{A\times\{L\}}) \otimes \pi_2^*(P|_{A\times\{L\}}) = m^*(P|_{A\times\{L\}})$$
$$\Leftrightarrow P|_{A\times\{L\}} \in \mathrm{Pic}^0(A)$$

$$\begin{tikzcd}
A \times \hat{A} \times A \arrow[r, "\tilde{m}"] \arrow[d, "\pi_{13}"'] \arrow[dr, phantom, "\lrcorner"] & A \times \hat{A} \arrow[d, "\pi_1"] \\
A \times A \arrow[r, "m"'] & A
\end{tikzcd}$$

Summary: Derived categories of coherent sheaves

For a (smooth projective) variety X we can associate the bounded derived category of coherent sheaves

$$X \rightsquigarrow \mathrm{D}^b(X)$$

Theorem (Bondal-Orlov 2001)

Let X, Y smooth projective varieties & with ample (anti) canonical bundle $\omega_X^{\pm 1}$. If $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$, then $X \cong Y$.

Theorem (Mukai 1981)

Let A abelian variety (=smooth projective variety & connected algebraic group). Then $\mathrm{D}^b(A) \simeq \mathrm{D}^b(\hat{A})$, but for many $A \ncong \hat{A}$.

References

- Daniel Huybrechts - Fourier-Mukai Transforms in Algebraic Geometry, Clarendon Press (2006)
- Ulrich Bunke - Abelian Varieties and the Fourier Mukai transformations, Lecture Notes (2005)
- James S. Milne - Abelian Varieties, Lecture Notes (2008)

Example of $A \ncong \hat{A}$

Let X abelian variety of dim 2 with $X \cong \hat{X}$ (e.g. principally polarized) with $\mathrm{End}(X) = \mathbb{Z}$. Let $p \in X(\mathbb{C})$ closed point of order 2.

Then $A := X/\langle p \rangle$ abelian variety with $A \ncong \hat{A}$.

As $\deg(X \to A \cong \hat{A} \to \hat{X} \cong X)$ has degree 4, but must be multiplication with some $n \in \mathbb{Z}$ which has degree n^4. Contradiction!