Algebraic Geometry: A Journey Through History

Towards Schemes

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Some notions of variety

- affine variety
- projective variety
- abstract varieties of A. Weil

Zariski topology

- Zariski (1944): introduced a topology on Zariski-Riemann spaces

$$V = \{\text{0-dim valuations on } K, \text{ vanishing on } k^\times\},$$

  k field, K function field over k

$\rightsquigarrow$ Zariski topology

- By 1950: A. Weil translated this to his abstract varieties

Oscar Zariski (1969)[1]

[1] Credit: George M. Bergman, GNU Free Documentation License, Version 1.2

Sheaves

- Leray (1946): concept of a sheaf and sheaf cohomology
- main idea of a sheaf: way of attaching data to the open sets of a topological space, such that gluing is possible
- Leray originally defined a sheaf as something that attaches data to a closed subset
- sheaf cohomology was defined for a specific sheaf, but Leray realized that it accepted all sheaves as coefficients

Jean Leray (1961)[2]

[2] Credit: Konrad Jacobs, CC BY-SA 2.0 DE

Sheaves

- important example of a sheaf: structure sheaf $\mathcal{O}_M$ for a complex manifold M

$$\mathcal{O}_M(U) = \{f\colon U \to \mathbb{C} \text{ holomorphic}\}$$

- sheaf cohomology recovers earlier notions, e. g. if M smooth complex projective curve, then it has genus

$$g = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M)$$

- sheaf cohomology recovers singular cohomology:

$$H^q_{\text{sing}}(M, A) = H^q_{\text{sh}}(M, \underline{A})$$

Sheaves on varieties

- Serre (1955): transferred the concept of sheaves to abstract varieties of Weil
- made definition of abstract variety (over an algebraically closed field k) much easier, using the concept of ringed space by Cartan
  - affine variety over k is a locally ringed space $(X, \mathcal{O}_X)$, where $X \subset k^n$ Zariski closed and $\mathcal{O}_X$ sheaf of regular functions on X,

$$\mathcal{O}_X(U) = \{f \colon U \to k \text{ regular}\}$$

$\rightsquigarrow$ variety is a locally ringed space glued by affine varieties

- Serre computed sheaf cohomology for $\mathcal{O}_X$-modules
  $\rightsquigarrow$ e. g. Serre-duality

Varieties and k-algebras

- until then: no intrinsic notion of affine variety without $\subset k^n$
- Hilbert Nullstellensatz (1893): for $k = \overline{k}$, have bijection

$$\begin{aligned} k^n &\overset{1:1}{\longleftrightarrow} \{\mathfrak{m} \subset k[x_1, ..., x_n] \text{ maximal ideal}\} \\ a = (a_1, ..., a_n) &\longmapsto (x_1 - a_1, ..., x_n - a_n) \end{aligned}$$

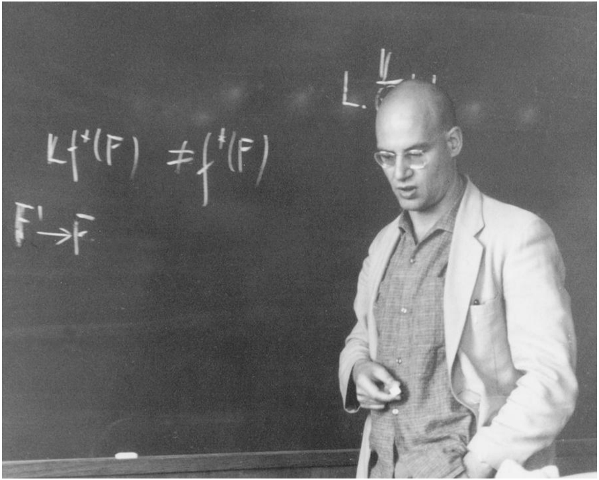
- more generally: $V \subset k^n$ affine variety, $R(V) := \{f|_V,\ f \in k[x_1, ..., x_n]\}$, then

$$V \overset{1:1}{\longleftrightarrow} \{\mathfrak{m} \subset R(V) \text{ maximal ideal}\}$$

- equivalence of categories

$$\{\text{affine varieties}\} \cong \{\text{reduced finitely generated } k\text{-algebras}\}^{\text{op}}$$

Affine schemes



Alexander Grothendieck 1965 in Bonn[3]

[3] Credit: Friedrich Hirzebruch, picture taken from CSG - Grothendieck Institute

Affine schemes

- Grotendieck (1957):

$$\begin{array}{ccc} \{\text{affine varieties}\} & \xrightarrow{\cong} & \{\text{reduced f. g. } k\text{-algebras}\}^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow \\ ? & \xrightarrow{\cong} & \{\text{Rings}\}^{\mathrm{op}} \end{array}$$

- needs modification, because for $\phi : A \to B$ ring homomorphism and $\mathfrak{m} \subset B$, $\phi^{-1}(\mathfrak{m})$ not maximal in general
- BUT: $\phi^{-1}(\mathfrak{p})$ prime for $\mathfrak{p} \subset B$ prime!

$\rightsquigarrow$ replace $V = \mathrm{Max}(A)$ by $V = \mathrm{Spec}(A) = \{\mathfrak{p} \subset A \text{ prime ideal}\}$

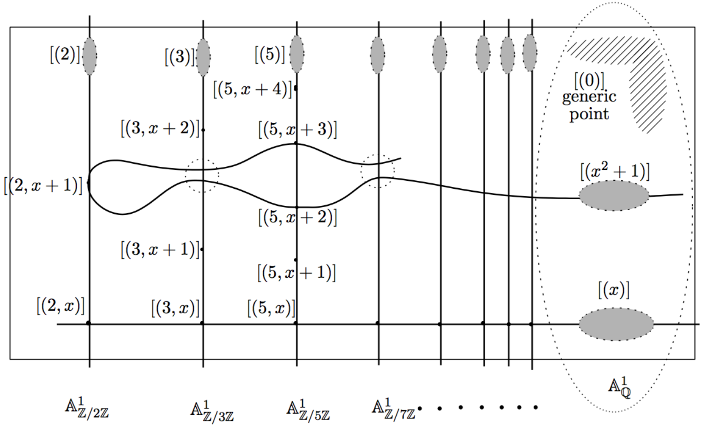
$\rightsquigarrow$ $(V, \mathcal{O}_V)$ affine scheme ($\mathcal{O}_V$ defined as before)

Affine schemes

$$\begin{array}{ccc}
\{\text{affine varieties}\} & \xrightarrow{\cong} & \{\text{reduced f. g. } k\text{-algebras}\}^{\mathrm{op}} \\
\big\downarrow & & \big\downarrow \\
\{\text{affine schemes}\} & \xrightarrow{\cong} & \{\text{Rings}\}^{\mathrm{op}}
\end{array}$$

Schemes

- get schemes by gluing affine schemes
- several notions generalize to this language, e.g. the generic point of an integral affine scheme $\mathrm{Spec}(A)$ is the zero ideal $0 \subset A$
- but also new notions, e. g. reduced scheme
- more flexibility, e. g. schemes over non-algebraically closed fields



Visualization of Spec($\mathbb{Z}[x]$)[4]

[4]Credit: David Mumford and Tadao Oda, image taken from Algebraic Geometry II (a penultimate draft)