Splitting Problems for Vector Bundles. Part III

Splitting algebraic vector bundles

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Homotopy classification of topological vector bundles

Theorem

Let T be a paracompact topological space (e.g., a CW complex). Then there are natural bijections

$$\mathcal{V}_r^{\mathbb{R}}(T) \cong [T, Gr_r(\mathbb{R})]$$

and

$$\mathcal{V}_r^{\mathbb{C}}(T) \cong [T, Gr_r(\mathbb{C})]$$

for all $r \geq 0$. In particular, if T is contractible, then all real/complex topological vector bundles over T are trivial.

Corollary

Let T be a paracompact topological space (e.g., a CW complex). Then the maps

$$\mathcal{V}_r^{\mathbb{K}}(T) \rightarrow \mathcal{V}_r^{\mathbb{K}}(T \times [0,1])$$

are bijections for all $r \geq 0$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Algebraic vector bundles over affine spaces

$\mathbb{A}^n := Spec(\mathbb{Z}[t_1, ..., t_n])$ affine spaces

$\mathbb{A}^n_S := \mathbb{A}^n \times S$ affine spaces over S (for S a scheme)

Notation. $\mathbb{A}^n_R := \mathbb{A}^n \times S$ for an affine scheme $S = Spec(R)$

Serre's problem (1955)

Let k be a field. Are all algebraic vector bundles over $\mathbb{A}^n_k$ trivial?

Theorem (Quillen, Suslin, 1976)

Let k be a field. Then all algebraic vector bundles over $\mathbb{A}^n_k$ are trivial.

$\mathbb{A}^1$-invariance for vector bundles

Bass-Quillen conjecture (1972/1976)

Let X be a regular Noetherian affine scheme of finite Krull dimension. Then the maps

$$\mathcal{V}_r(X) \to \mathcal{V}_r(X \times \mathbb{A}^1)$$

are bijections for all $r \geq 0$.

Theorem (Lindel, 1982)

The Bass-Quillen conjecture holds for any smooth affine scheme X over $Spec(k)$, where k is a field.

Theorem (Popescu, 1989)

The Bass-Quillen conjecture holds for any smooth affine scheme X over $Spec(\mathbb{Z})$.

$\mathbb{A}^1$-homotopy theory or motivic homotopy theory

S - a quasi-compact and quasi-separated base scheme
(e.g., $S = Spec(k)$ for some field k or $S = Spec(\mathbb{Z})$)

Sm_S - category of smooth S-schemes

Now embed this category into a larger category called

Spc_S - the category of spaces over S
(the category of simplicial Nisnevich sheaves of sets over Sm_S)

Formally invert all the projections $\mathcal{X} \times \mathbb{A}^1_S \to \mathcal{X}$ for any space $\mathcal{X}$.

The category obtained in this form is called

$\mathcal{H}(S)$ - the unstable $\mathbb{A}^1$-homotopy category over S

The unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$

Objects of $\mathcal{H}(S)$ are spaces (i.e., the objects of the category Spc_S)

Morphisms in $\mathcal{H}(S)$ are interpreted as **homotopy classes**

Notation. For spaces $\mathcal{X}, \mathcal{Y}$, we let $[\mathcal{X}, \mathcal{Y}] := Hom_{\mathcal{H}(S)}(\mathcal{X}, \mathcal{Y})$.

Let $i_0 : S \hookrightarrow \mathbb{A}^1_S, s \mapsto (0, s)$ and $i_1 : S \hookrightarrow \mathbb{A}^1_S, s \mapsto (1, s)$.

Definition

Let $X, Y \in Sm_S$. Two morphisms $f, g : X \to Y$ are called naively $\mathbb{A}^1$-homotopic if there is a morphism $H : X \times \mathbb{A}^1_S \to Y$ with $H \circ (id_X \times i_0) = f$ and $H \circ (id_X \times i_1) = g$.

One obtains a well-defined map $Hom_{Sm_S}(X, Y)/_{\sim_{naive}} \to [X, Y]$.

Caveat: This map is not bijective in general!

Affine representability for algebraic vector bundles

Theorem (Morel, Schlichting, Asok-Hoyois-Wendt)

Let S be either $Spec(k)$ for some field k or $Spec(\mathbb{Z})$ and $r \geq 0$. Then there are natural bijections

$$\mathcal{V}_r(X) \cong [X, Gr_r]$$

for all **affine** $X \in Sm_S$.

- Morel (2006): $r \neq 2$, $S = Spec(k)$, k perfect field
- Schlichting (2015): arbitrary r, $S = Spec(k)$, k perfect field
- Asok-Hoyois-Wendt (2015): as above (in essence, theorem is equivalent to the Bass-Quillen conjecture for all **affine** $X \in Sm_S$)

The splitting problem via $\mathbb{A}^1$-homotopy theory

Let k be a perfect field and X a smooth affine k-scheme.

The splitting problem for algebraic vector bundles

When does an algebraic vector bundle $\mathcal{E}$ over X of rank r admit an isomorphism $\mathcal{E} \cong \mathcal{E}' \oplus \underline{1}$ for some vector bundle $\mathcal{E}'$?

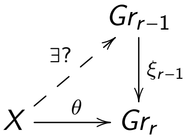
In other words: When does $[\mathcal{E}]$ lie in the image of the map $s_{r-1} : \mathcal{V}_{r-1}(X) \to \mathcal{V}_r(X), [\mathcal{E}'] \mapsto [\mathcal{E}' \oplus \underline{1}]$?

The inclusion $\xi_{r-1} : Gr_{r-1} \hookrightarrow Gr_r$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{V}_{r-1}(X) & \xrightarrow{s_{r-1}} & \mathcal{V}_r(X) \\ \downarrow \cong & & \downarrow \cong \\ [X, Gr_{r-1}] & \xrightarrow{(\xi_{r-1})_*} & [X, Gr_r]. \end{array}$$

The splitting problem as a lifting problem

The splitting problem reduces to the following lifting problem in $\mathcal{H}(k)$:



There exists a version of **Moore-Postnikov factorizations** in $\mathbb{A}^1$-homotopy theory.

The morphism of spaces $\xi_{r-1} : Gr_{r-1} \to Gr_r$ can be factored as a sequence of simpler lifting problems.

Each of these simpler lifting problems can be reduced to studying cohomology.

The obstruction theory for ξ_{r-1}

The morphism ξ_{r-1} fits into a fiber sequence

$$\mathbb{A}^r_k \setminus 0 \to Gr_{r-1} \xrightarrow{\xi_{r-1}} Gr_r$$

measuring the failure of ξ_{r-1} to be an isomorphism in $\mathcal{H}(k)$.

For a space $\mathcal{X}$ with a chosen basepoint $x : Spec(k) \to \mathcal{X}$, one defines $\mathbb{A}^1$-homotopy sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x) : Sm_k^{op} \to Sets$ for $i \geq 0$.

For $i = 1$, these are sheaves of groups, for $i \geq 2$ sheaves of abelian groups.

Using the motivic Moore-Postnikov factorization, one obtains for $i \geq 0$ cohomological obstructions

$$o_i \in H^{i+1}_{Nis}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}^r_k \setminus 0, *)(\det \theta))$$

for a lift of $\theta : X \to Gr_r$ along ξ_{r-1}.

Theorem (Morel, 2006)

For $r \geq 2$, one has $\pi_i^{\mathbb{A}^1}(\mathbb{A}_k^r \setminus 0, *) \cong \begin{cases} 0 & \text{if } i \leq r-2 \\ \mathbf{K}_r^{MW} & \text{if } i = r-1. \end{cases}$

Theorem

Let X be a smooth affine k-scheme of dimension $d \geq 1$. Any vector bundle $\mathcal{E}$ over X of rank $r > d$ admits a decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \underline{1}$.

Proof.

All obstructions o_i are zero, as $H_{Nis}^{i+1}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}_k^r \setminus 0, *)(\det \theta)) = 0$

- for $i \geq d$ by dimension reasons
- for $i < d$ by Morel's computations above.

□

Motivic Euler classes

Theorem (Morel, 2006)

For $r \geq 2$, one has $\pi_i^{\mathbb{A}^1}(\mathbb{A}_k^r \setminus 0, *) \cong \begin{cases} 0 & \text{if } i \leq r-2 \\ \mathbf{K}_r^{MW} & \text{if } i = r-1. \end{cases}$

Theorem (Morel, 2006)

Let X be a smooth affine k-scheme of dimension $d \geq 2$. A vector bundle $\mathcal{E}$ over X of rank d admits a decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \underline{1}$ if and only if the **Euler class** $o_{d-1} \in H_{Nis}^d(X, \mathbf{K}_d^{MW}(\det\theta))$ is zero.

Proof.

$H_{Nis}^{i+1}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}_k^d \setminus 0, *)(\det\theta)) = 0$

- for $i \geq d$ by dimension reasons
- for $i < d-1$ by Morel's computations.

□

Thank you!