The Hasse–Weil zeta function for groups

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Max Gheorghiu
HHU Düsseldorf

Idea

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The variety V is defined over a set of equations. For almost all primes p one can take the equations modulo p and define

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The Hasse–Weil zeta function of V is defined as

$$\zeta_V : U \subseteq \mathbb{C} \to \mathbb{C}, \quad s \mapsto \prod_{p \text{ prime}} \zeta_{V,p}(s)$$

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RIEMANN ZETA FUNCTION

If $V = \{*\}$ is a point, then

$$\zeta_{V,p}(s) = \frac{1}{1 - p^{-s}}$$

and thus, its Hasse–Weil zeta function is the Riemann zeta function

$$\zeta_V(s) = \zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}.$$

HASSE–WEIL CONJECTURE

Any Hasse–Weil zeta function $\zeta_V(s)$ extends meromorphically to the entire complex plane $\mathbb{C}$ and satisfies a functional equation similar to that of the Riemann zeta function $\zeta(s)$.

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Millenium Problem: Birch and Swinnerton-Dyer conjecture

- The number of infinite-order generators for an elliptic curve E (Prüfer rank) is the order of the zero of the Hasse–Weil zeta function $\zeta_E(s)$ at $s = 1$ and
- ...

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Hasse–Weil zeta function (groups)

For $n \geq 1$ let

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The Hasse–Weil zeta function of G is defined as

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where

$$\zeta_{G,p}(s) := \exp\left(\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{r_n(G, \mathbb{F}_{p^j})}{j} p^{-sjn} \underbrace{|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|}_{\text{scaling factor}} \right).$$

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HASSE–WEIL ZETA FUNCTION FOR $\mathbb{Z}$

If $G = \mathbb{Z}$, then it has $p^j - 1$ one-dimensional absolutely irreducible representations over $\mathbb{F}_{p^j}$. Its Hasse–Weil zeta function is

$$\zeta_{\mathbb{Z}}(s) = \prod_{p \text{ prime}} \exp\left(\sum_{j=1}^{\infty} \frac{p^j - 1}{j} p^{-sj} \right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \left(1 - \frac{1}{p^s}\right)$$

$$= \frac{\zeta(s-1)}{\zeta(s)}.$$

Profinite groups and completions

- A profinite group is a topological group that can be assembled from finite groups.
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ZETA FUNCTIONS VIA PROFINITE GROUPS

The Hasse–Weil zeta function $\zeta_G(s)$ of a group G can be defined via its profinite completion $\widehat{G}$. Hasse–Weil zeta functions can be defined for every profinite group.

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Theorem (Corob Cook, Kionke, Vannacci, '24)

The reciprocal value $\zeta_G(k)^{-1}$ at a sufficiently large integer k coincides with the probability that k random elements generate the completed group ring $\widehat{\mathbb{Z}}[[G]]$ of a profinite group G.

Questions?