Interlude:

Projective Modules

R commutative ring

All modules finitely generated

Def: F **free** module$/R$: $\quad F \cong R^{\oplus n}$

P **projective** module$/R$:

$\exists Q, n$: $\quad P \oplus Q \cong R^{\oplus n}$

Eg:

(a) projective modules$/\mathbb{Z}$: free (as $P \leq \mathbb{Z}^{\oplus n}$)

PID R

Def: F free module$/R$: $\quad F \cong R^{\oplus n}$

P projective module$/R$:

$\exists Q, n$: $\quad P \oplus Q \cong R^{\oplus n}$

Eg:

(b) projective module$/$local ring is free

proof: Take n minimal s.t.

$$\begin{tikzcd} \underset{\text{kernel}}{Q} \arrow[r, hook] & R^{\oplus n} \arrow[r, two heads] & P \arrow[l, dashed, bend right] \end{tikzcd}$$

Projectivity of P implies

$$P \oplus Q \cong R^{\oplus n}$$

Nakayama's lemma now implies:

$(\text{minimal } \#\text{generators of } Q) = 0$, i.e. $Q = 0$. $\square$

Def: F free module$/R$: $\quad F \cong R^{\oplus n}$

P projective module$/R$:

$\exists Q, n$: $\quad P \oplus Q \cong R^{\oplus n}$

Eg:

(c) $R \times 0$ projective $/ \; R \times R$

$$R \times 0 \oplus 0 \times R \cong (R \times R)^{\oplus 1}$$

(d) $R := \mathbb{R}[x,y,z] \big/ x^2+y^2+z^2=1$

$$P \rightarrowtail R^{\oplus 3} \xrightarrow[(x\ y\ z)]{} \twoheadrightarrow R$$

$P \oplus R \cong R^{\oplus 3} \quad$ (but $P \not\cong R^{\oplus 2}$)

Def: F free module $/R$: $\quad F \cong R^{\oplus n}$

P projective module $/R$:

$\exists Q, n$: $\quad P \oplus Q \cong R^{\oplus n}$

Eg:

(e) projective $/ \mathbb{C}[x]$ — free

projective $/ \mathbb{C}[x,y]$?

k a field

Thm 1 (Quillen–Suslin):
Every projective module over $k[x_1, \dots, x_d]$ is free.

Def: **rank** of projective P / integral domain R with field of fractions F

$:= \dim_F (P \otimes_R F)$

Here, $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus(\operatorname{rank} P)}$ for all $\mathfrak{p} \subseteq R$.

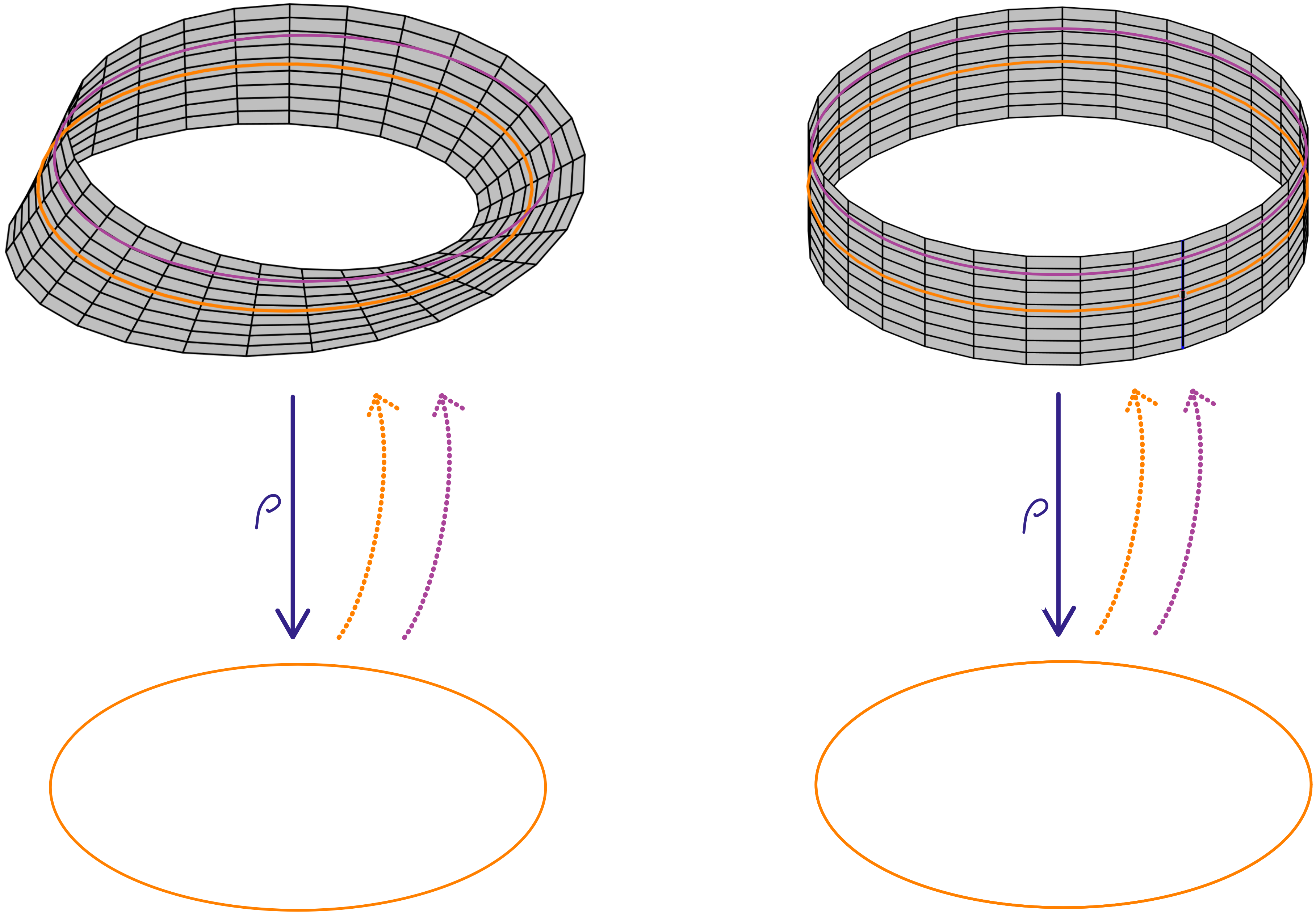
Thm 2: R integral k-algebra, smooth $/k$, of Krull dimension d (≥ 1).

P projective $/R$ of rank r.

If $r > d$, then $P \cong P' \oplus R$.

Projective modules are vector bundles

modules of sections of



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Projective modules are vector bundles

Thm A (Swan): X finite-dimensional connected CW complex.

$$\left(\text{real vector bundles}/X\right) \xleftrightarrow{1:1} \left(\text{projective modules}/R\right)$$

$$R := C^0(X; \mathbb{R})$$

$$\begin{array}{c} E \\ \downarrow{\scriptstyle p} \\ X \end{array} \longmapsto \{\text{continuous sections of } p\}$$

$$\shortparallel$$

$$\Gamma_{\mathcal{C}}(E)$$

Projective modules are vector bundles

Thm A (Swan): X finite-dimensional connected CW complex.

$$\left(\begin{matrix}\text{real vector} \\ \text{bundles} / X\end{matrix} \right) \xleftrightarrow{1:1} \left(\begin{matrix}\text{projective} \\ \text{modules} / R\end{matrix} \right) \qquad R := C^0(X; \mathbb{R})$$

$$E \longmapsto \Gamma S(E)$$

proof:

$$\underline{1} \longmapsto \Gamma S(\underline{1}) \cong R$$

$$\underline{n} \longmapsto \Gamma S(\underline{n}) \cong R^{\oplus n}$$

So

$$\left(\begin{matrix}\text{trivial vector} \\ \text{bundles} / X\end{matrix} \right) \xleftrightarrow{1:1} \left(\begin{matrix}\text{free} \\ \text{modules} / R\end{matrix} \right)$$

Projective modules are vector bundles

Thm A (Swan): X finite-dimensional connected CW complex.

$$\left(\begin{array}{c}\text{real vector}\\ \text{bundles}/\\ X\end{array}\right) \xleftrightarrow{1:1} \left(\begin{array}{c}\text{projective}\\ \text{modules}/\\ R\end{array}\right) \qquad R := C^0(X;\mathbb{R})$$

$$E \longmapsto \Gamma\mathcal{S}(E)$$

proof:

Also, $\Gamma\mathcal{S}(E)$ projective:

Assumptions on X imply $E \oplus F \cong \underline{n}$,

hence $\Gamma\mathcal{S}(E) \oplus \Gamma\mathcal{S}(F) \cong R^{\oplus n}$

[...] $\square$

Projective modules are vector bundles

Thm B (Serre): $X = \mathrm{Spec}(A)$ connected affine variety $/k$.

replace "continuous" by algebraic

$$\begin{pmatrix} \text{vector} \\ \text{bundles}/ \\ X \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} \text{projective} \\ \text{modules}/ \\ A \end{pmatrix}$$

$$\begin{array}{ccc} E & & \\ \pi \downarrow & \longmapsto & \Gamma S(E) \\ X & & \end{array}$$

replace "continuous" by algebraic

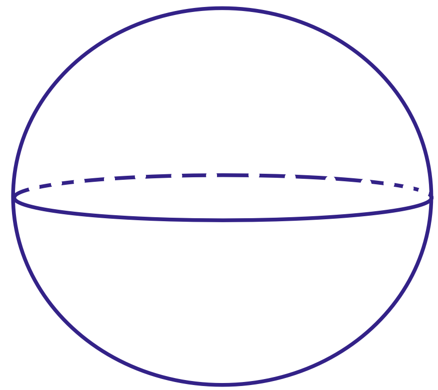
Eg:

(d) $R := \mathbb{R}[x,y,z] \big/ x^2+y^2+z^2=1$ ← coordinate ring of real algebraic sphere

$$\begin{tikzcd} P \arrow[r, tail] & R^{\oplus 3} \arrow[r, two heads, "(x\ y\ z)"'] & R \end{tikzcd}$$

(algebraic) sections of tangent bundle

(algebraic) sections of trivial bundle



Algebraic sphere, tangent bundle / $\mathbb{R}$ "realize" to continuous sphere, tangent bundle.

You cannot comb a hedgehog.

So $P \not\cong R \oplus Q$ for any Q.

Projective modules are vector bundles

Thm B (Serre): $X = \mathrm{Spec}(A)$ connected affine variety $/k$.

$$\left(\begin{array}{c}\text{vector}\\ \text{bundles}/\\ X\end{array}\right) \xleftrightarrow{1:1} \left(\begin{array}{c}\text{projective}\\ \text{modules}/\\ A\end{array}\right)$$

$$E \longmapsto \Gamma\mathcal{S}(E)$$

Over general X (not necessarily affine) with structure sheaf $\mathcal{O}_X$:

$$\left(\begin{array}{c}\text{vector}\\ \text{bundles}/\\ X\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{locally free}\\ \text{coherent}\\ \text{sheaves}\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{modules}/\\ \Gamma\mathcal{O}_X\end{array}\right)$$

Over general X (not necessarily affine) with structure sheaf $\mathcal{O}_X$:

$$\left(\begin{matrix}\text{vector}\\ \text{bundles}/\\ X\end{matrix}\right) \xleftrightarrow[\mathcal{S}]{1:1} \left(\begin{matrix}\text{locally free}\\ \text{coherent}\\ \text{sheaves}\end{matrix}\right) \xrightarrow{\Gamma} \left(\begin{matrix}\text{modules}/\\ \Gamma\mathcal{O}_X\end{matrix}\right)$$

not useful in non-affine context

Eg: $\underline{1} \;\hat{=}\; \mathcal{O}_X$

tautological line bundle/ $\mathbb{R}P^n$ $\;\hat{=}\; \mathcal{O}(-1) \longmapsto 0$

Thm 1 (Quillen–Suslin): Every vector bundle over $\mathbb{A}^d_k$ is trivial.

Thm 2: X smooth affine variety $/k$, of dimension d (≥ 1).
E vector bundle $/X$ of rank r.
If $r > d$, then $E \cong E' \oplus \underline{1}$.

Thm 3: X, E as above, $d \geq 2$, $r = d$.

$$E \cong E' \oplus \underline{1} \iff o_{d-1}(E) \in H^d_{Nis}(X, K^{MW}(\det E)) \text{ vanishes}$$