Previous Talk

Given: $f \in \mathbb{Z}[x_1, \dots, x_d]$

p-adic measure

Poincaré series: $Q_f(T) = \sum_{m \geq 0} \mu(X_m) \underbrace{p^{-dm} \cdot T^m}_{=(p^{-d}T)^m}$

where $X_m = \{\underline{x} \in \mathbb{Z}_p^d \mid v(f(\underline{x})) \geq m\}$

$\boxed{\text{Thm: } \cancel{Q_f(T)}\ \tilde{Q}_f(T) \in \mathbb{Q}(T)}$

This Talk:

Generalize this.

$\tilde{Q}_f(T) := Q_f(p^d T) = \sum_{m \geq 0} \mu(X_m) T^m$

Note: Can remove p^{-dm}

$\boxed{\text{Thm: Given "suitable" } X_m \subset \mathbb{Z}_p^d \ (m \in \mathbb{N}):}$

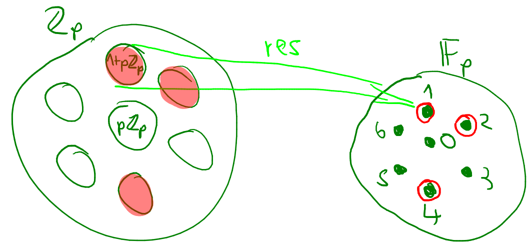
set $Q_{X_\bullet}(T) := \sum_{m \geq 0} \mu(X_m) T^m$

Then $Q_{X_\bullet}(T) \in \mathbb{Q}(T)$

Example

$\boxed{p \neq 2}$

$$Q_{X_0}(T) = \sum_{m \geq 0} \mu(X_m)\, T^m \quad \text{where } X_m = \{x \in \mathbb{Z}_p \mid x \text{ is a square in } \mathbb{Q}_p,\ v(x) = m\}$$



- $X_0 = \{x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p \mid x \text{ square in } \mathbb{Q}_p\}$

$$\Updownarrow$$

$$\exists y: \quad x = y^2$$

$$\Updownarrow \quad \text{Hensels Lemma}$$

$$\exists\, res(y): \quad res(x) = res(y)^2$$

$$\Updownarrow$$

$$res(x) \text{ square in } \mathbb{F}_p$$

$$\mu(X_0) = \frac{p-1}{2} \cdot \frac{1}{p}$$

- $X_1 = \emptyset \qquad (x = y^2 \Rightarrow v(x) = 2v(y) \ ↯)$
- $X_{2l+1} = \emptyset \qquad \mu(X_{2l+1}) = 0$
- $X_{2l} = \{p^2 \cdot x \mid x \in X_0\} \qquad \mu(X_{2l}) = \frac{p-1}{2p} \cdot p^{-2l}$
- $Q_{X_0}(T) = \sum_{l \geq 0} \frac{p-1}{2p} \cdot p^{-2l} \cdot T^{2l} = \frac{p-1}{2p} \cdot \sum_{l \geq 0} (p^{-2} \cdot T)^l = \frac{p-1}{2p} \cdot \frac{1}{1 - p^{-2} T} \in \mathbb{Q}(T)$

General case: Condition on $X_\bullet$?

Ex 1: $X_m = \{ \underline{x} \in \mathbb{Z}_p^d \mid v(f(\underline{x})) \geq m \}$

Ex 2: $X_m = \{ x \in \mathbb{Z}_p \mid \underbrace{x \text{ is square in } \mathbb{Q}_p}_{\exists y:\, y^2 = x},\ v(x) = m \}$

Thm: $Q_{X_\bullet}(T) := \sum_{m \geq 0} \mu(X_m)\, T^m \in \mathbb{Q}(T)$ if

$X_\bullet$ is given by a ? 1st order formula in the language of valued fields ? ? ? ? ? ?

1st order fmla in the lang. of val. flds

Ex 1: "$x_1,\dots,x_d\in\mathbb{Z}_p \wedge \underbrace{v(f(x_1,\dots,x_d))\geq m}_{??}$" where $f\in\mathbb{Z}[x_1,\dots,x_d]$

Ex 2: "$\exists y: y^2=x \wedge v(x)=m$" (in x and m (but not y!))

Defn: A 1st order fmla in the language of valued fields is obtained as follows:

- Write $f(x_1,\dots,x_d)=0$ for $f\in\mathbb{Z}[x_1,\dots,x_d]$ (in $x_1,\dots,x_d$) $X_m=\{\underline{x}\in\mathbb{Q}_p^d \mid f(\underline{x})=0\}$

  or $v(x)\geq 0$ (in x) $X_m=\mathbb{Z}_p$

  or $v(x)=m$ (in x and m) $X_m=\{x\in\mathbb{Q}_p \mid v(x)=m\}$
- Apply boolean combinations ($\wedge$, $\vee$, $\neg$) — $\wedge$ intersection, $\vee$ union, $\neg$ complement
- Apply quantifiers: $\exists x$, $\cancel{\forall x}$ — projection ($\exists x$ means $\exists x\in\mathbb{Q}_p$)

Every such fmla defines a family of sets X_m

Note: One should really say: "formula in $\overbrace{\text{variables}}^{\text{valued field}}$ $x_1,\dots,x_d$" $\rightsquigarrow X_m\subset\mathbb{Q}_p^d$

"...and in the value group variable m" $\rightsquigarrow X_\bullet\subset\mathbb{Z}_p^d\times\mathbb{Z}$ $\quad(X_m=\{\underline{x}\in\mathbb{Q}_p^d \mid (\underline{x},m)\in X_\bullet\})$

1st order fmlas are very flexible

Defn: consists of $f(\underline{x}) = 0$

$v(x) \geq 0$

$v(x) = m$

$\wedge, \vee, \neg$

$\exists x$

minimalistic definition

Can also express: ...namely as follows:

Can also express:	...namely as follows:
$\forall x : \text{blah}$	$\neg \exists x : \neg\, \text{blah}$
$v(f(\underline{x})) \geq 0$	$\exists y : f(\underline{x}) = y \wedge v(y) \geq 0$
$v(f(\underline{x})) \geq v(g(\underline{x}))$	$\exists y : f(\underline{x}) = y \cdot g(\underline{x}) \wedge v(y) \geq 0$ $\big(v(f(\underline{x})) = v(y) + v(g(\underline{x}))\big)$
$v(f(\underline{x})) = v(g(\underline{x}))$	$v(f(\underline{x})) \geq v(g(\underline{x})) \wedge v(g(\underline{x})) \geq v(f(\underline{x}))$
Ex 1: √ — $v(f(\underline{x})) \geq m$	$\exists y : v(y) = m \wedge v(f(\underline{x})) \geq v(y)$
$\exists m' \in \mathbb{Z} : \text{blah}(m')$ — val grp var	$\exists y : y \neq 0 \wedge \text{blah}(v(y))$
etc.	

Proof & Applications

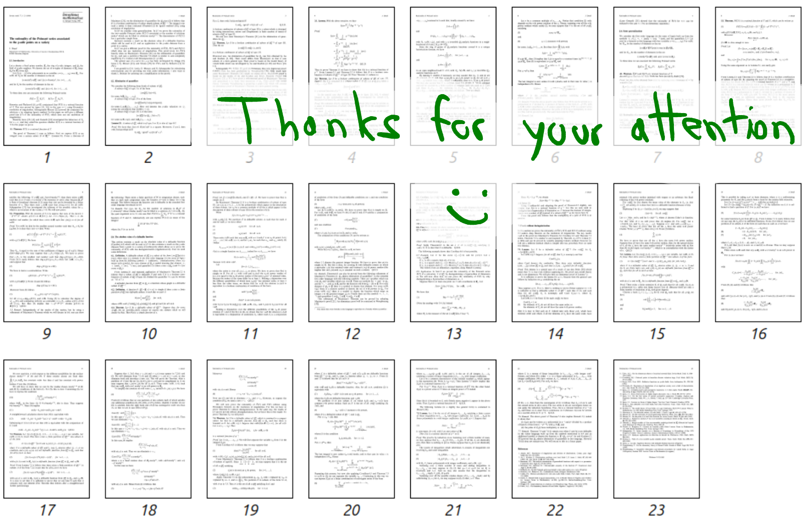
Thm: $Q_{X_\bullet}(T) := \sum_{m \geq 0} \mu(X_m) T^m \in \mathbb{Q}(T)$ if

$X_\bullet$ is given by a 1st order formula in the language of valued fields

Example application:

$\mu(X_m) = \#$ subgps of $GL_n(\mathbb{Z}_p)$ of index p^m

Proof:



Thanks for your attention!