Splitting Problems for Vector Bundles. Part I

Splitting topological vector bundles

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Definition of (real topological) vector bundles

Fix a topological space B.

Example

The trivial bundle $\underline{r}$ of rank r over B is $\underline{r} := (B \times \mathbb{R}^r \xrightarrow{\mathrm{pr}_1} B)$.

Definition

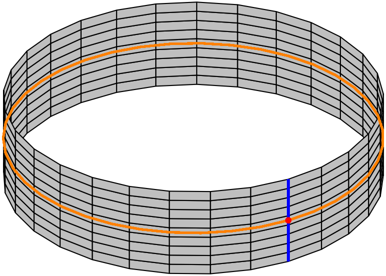
A (real topological) vector bundle $\xi^r = (E \xrightarrow{\rho} B)$ over B of rank r is given by a topological space E and a surjective map $\rho\colon E \to B$ such that

- $\rho^{-1}(b)$ is a r-dim $\mathbb{R}$-vector space for all $b \in B$; and
- for each $b \in B$ there is an open neighborhood $U \subset B$ of b and an isomorphism $\psi\colon U \times \mathbb{R}^r \xrightarrow{\sim} \rho^{-1}(U)$ with $\rho \circ \psi = \mathrm{pr}_1$ such that the restrictions to the fibers

$$\{b'\} \times \mathbb{R}^r \xrightarrow{\sim} \rho^{-1}(b')$$

are isomorphisms of $\mathbb{R}$-vector spaces for all $b' \in U$.

Consider for $B := \mathbb{S}^1$



$E := \mathbb{S}^1 \times \mathbb{R}^1$

$B := \mathbb{S}^1$

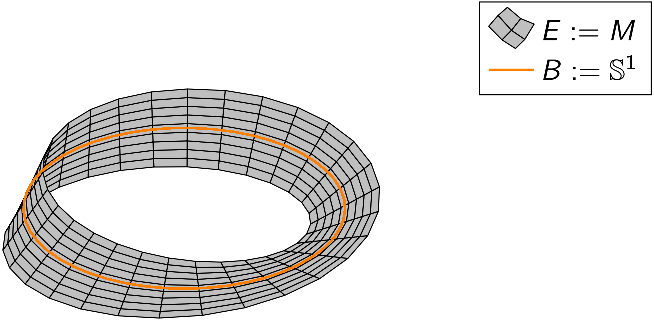
$b \in \mathbb{S}^1$

$pr_1^{-1}(b) = \{b\} \times \mathbb{R}^1 \cong \mathbb{R}^1$

the trivial vector bundle $\underline{1} := (\mathbb{S}^1 \times \mathbb{R}^1 \xrightarrow{pr_1} \mathbb{S}^1)$ over $\mathbb{S}^1$

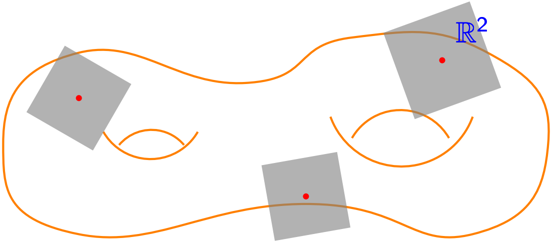
Examples of vector bundles over $B := \mathbb{S}^1$

Consider for $B := \mathbb{S}^1$



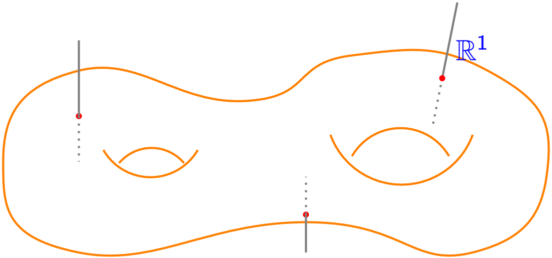
the Möbius strip $\mathbb{M}^1 := (M \xrightarrow{\rho} \mathbb{S}^1)$ over $\mathbb{S}^1$.
This is also the tautological line bundle over $\mathbb{RP}^1 \cong \mathbb{S}^1$. In general, glue the vector space X to $X \in \mathbb{RP}^n$ to get the tautological line bundle over $\mathbb{RP}^n$.

Examples of vector bundles over B



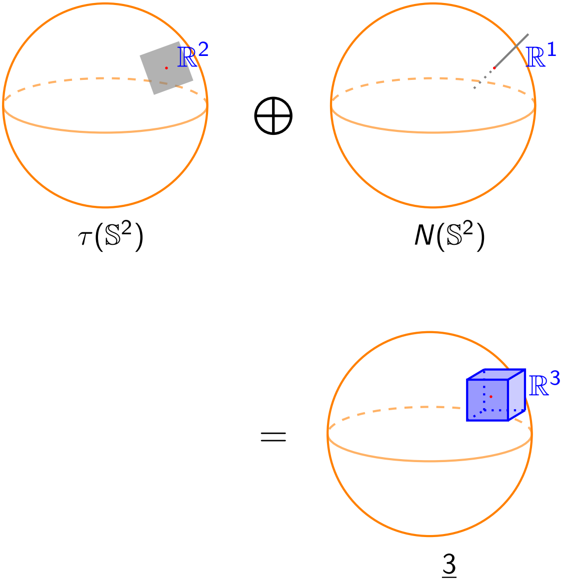
the tangential bundle $\tau(B)$ of B

Examples of Vector Bundles over B

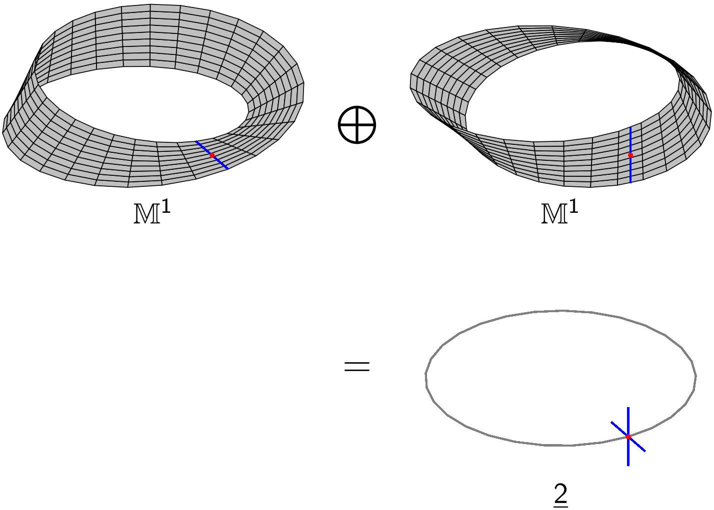


the normal bundle $N(B)$ of B

Whitney sum $\bigoplus$ (:= fiber product)



Whitney sum $\bigoplus$ (:= fiber product)



Goal: Classify vector bundles

- Fix B be a d-dim CW complex (e.g. $B = \mathbb{D}^d, \mathbb{S}^d, \mathbb{R}^d$).
- $\mathcal{V}_r(B) := \{\text{vector bundles of rank } r \text{ over } B\}_{/\cong}$
- Gr_r the Grassmanian of r-dim subspaces of $\mathbb{R}^\infty$

Theorem (Pontryagin–Steenrod representability theorem)

For $\xi^r \in \mathcal{V}_r(B)$ and $n > r + d$ there is $\xi^r \rightarrowtail \underline{n}$. Hence $\rho^{-1}(b) \subset \mathbb{R}^n \subset \mathbb{R}^\infty$, i.e. $\rho^{-1}(b) \in \mathrm{Gr}_r$. Furthermore, there is a bijection:

$$\mathcal{V}_r(B) \xrightarrow{\cong} [B, \mathrm{Gr}_r]$$

$$\xi^r = (E \xrightarrow{\rho} B) \mapsto \begin{bmatrix} B \to \mathrm{Gr}_r \\ b \mapsto \rho^{-1}(b) \end{bmatrix}$$

Example

$$\mathcal{V}_r(\mathbb{R}^d) = \{\underline{r}\}; \mathcal{V}_1(\mathbb{S}^1) = \{\underline{1}, \mathbb{M}^1\}; \mathcal{V}_{>1}(\mathbb{S}^1) = ???$$

The splitting problem for topological vector bundles

Question: Splitting Problem

When does a vector bundle ξ^r admit an isomorphism $\xi^r \cong \nu^{r-1} \oplus \underline{1}$ for some vector bundle ν^{r-1}?

In other words: When does ξ^r lie in the image of the map
$s_{r-1} \colon \mathcal{V}_{r-1}(B) \to \mathcal{V}_r(B), \quad \nu^{r-1} \mapsto \nu^{r-1} \oplus \underline{1}$?

Theorem

If $d < r$, then $s_{r-1} \colon \mathcal{V}_{r-1}(B) \to \mathcal{V}_r(B)$ is surjective.

Corollary

If $d \leq r$, then $\xi^r \cong \nu^d \oplus \underline{r-d}$ for some ν^d of rank d.

So classifying vector bundles over d-dim B reduces to classifying vector bundles of rank $\leq d$.

Example

$\mathcal{V}_r(\mathbb{S}^1) = \{\underline{r}, \mathbb{M}^1 \oplus \underline{r-1}\}$

Proof-sketch of Thm: If $d < r$, then s_{r-1} is surjective

Let $j\colon \mathrm{Gr}_{r-1} \to \mathrm{Gr}_r$, $V \mapsto V \oplus \mathbb{R}$, then the diagram

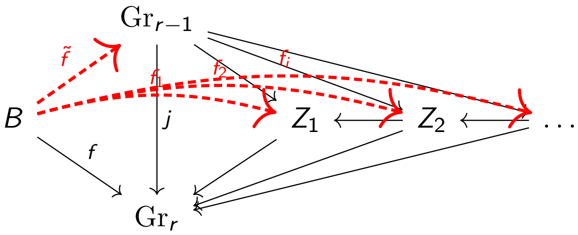
$$\begin{array}{ccc} \mathcal{V}_{r-1}(B) & \xrightarrow{s_{r-1}} & \mathcal{V}_r(B) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ [B, \mathrm{Gr}_{r-1}] & \xrightarrow{j_*} & [B, \mathrm{Gr}_r] \end{array}$$

commutes. Hence surjectivity translates into the existence of a lifting:

$$\begin{array}{ccc} & & \mathrm{Gr}_{r-1} \\ & \overset{\exists \tilde{f}}{\nearrow} & \big\downarrow{\scriptstyle j} \\ B & \xrightarrow{\forall f} & \mathrm{Gr}_r \end{array}$$

Proof-sketch of Thm: If $d < r$, then s_{r-1} is surjective

Take a Moore-Postnikov tower of principle fibrations of j



To lift f_i to f_{i+1} we need some $\omega_i \in H^{i+1}(B, \pi_i(\mathbb{S}^{r-1}))$ to be zero.

Fact:

If $d < r$, then $H^{i+1}(B, \pi_i(\mathbb{S}^{r-1})) = 0$ for all i.

- for $i + 1 \geq r > d$: by dimension reasons;
- for $i < r - 1$: $\pi_i(\mathbb{S}^{r-1}) = 0$.

Hence we get all the lifts f_i and then also a lift $\tilde{f}$.

And for $r \leq d$?

For example $r = d$:

$$H^{i+1}(B, \pi_i(\mathbb{S}^{r-1})) = \begin{cases} 0 & \text{, if } i+1 > r = d \\ 0 & \text{, if } i < r-1 \\ H^d(B, \pi_{r-1}(\mathbb{S}^{r-1})) & \text{, if } i = r-1 \end{cases}$$

Theorem

An (orientable) vector bundle ξ^d of rank $r = d$ over a d-dim B splits off a trivial bundle $\underline{1}$, if and only if, the Euler class $\omega_{d-1} \in H^d(B, \mathbb{Z})$ is zero.

Thank you!