Algebraic Geometry: A Journey Through History

Birth

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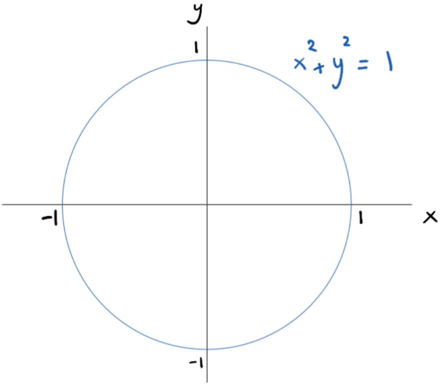
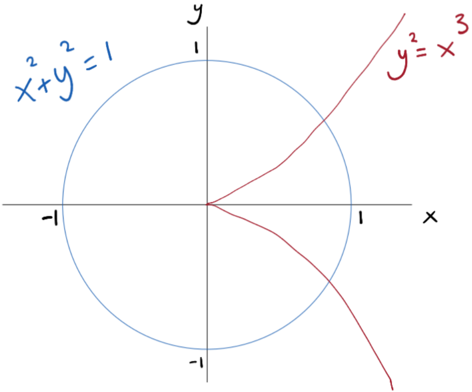
GRK 2240 Retreat, October 7th 2025

Motivation



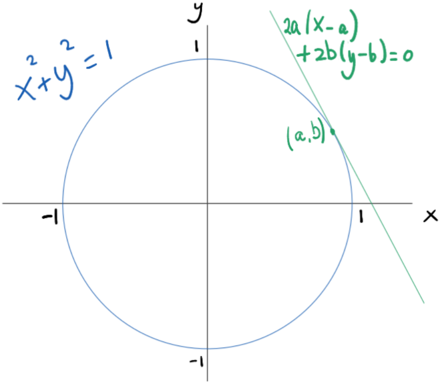
Question: What geometric properties can we derive from the equation that defines this geometric object?

Example 1: Smoothness



We can detect the non-smooth point in the red curve by partially deriving the equations!

Example 2: Tangents



Similarly, tangent lines are just given (algebraically) by the partial derivatives.

'Definition'

Algebraic geometry is the algebraic study of geometric objects through the polynomial equations that (locally) define them.

Why algebraic geometry?

There are some compelling reasons to do algebraic geometry:

- Algebra is fun!
- It provides a more precise framework to do geometry.
- Many geometrical objects arise from algebraic equations, think easy (lines, circles etc.) or more complicated (compact complex manifolds).

The presentation

Today, we give you a guided tour through the history of algebraic geometry. Up first: the birth of algebraic geometry, and that which came before. Some excellent references are:

- The Historical Development of Algebraic Geometry by J. Dieudonné;
- Mathematics and Its History (A Concise Edition) by J. Stillwell;
- Science Awakening I by B. L. van der Waerden.

Greek geometry (600-200 BC)

The mathematical basis

Definition

1. A ***point*** is that which has no part.
2. A ***line*** is breadthless length.
3. A ***circle*** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
4. A ***number*** is a multitude composed of units.

Proposition

If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

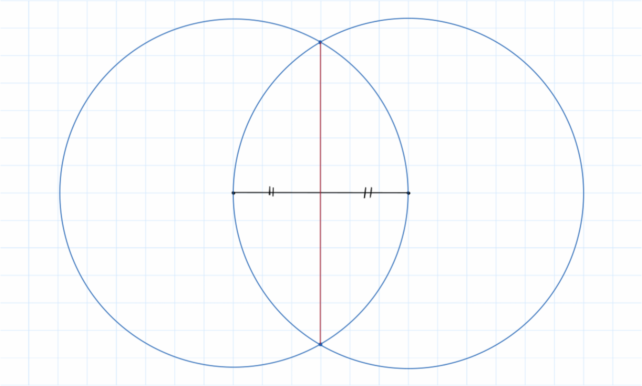
Geometric algebra

Instead of algebraic geometry, the Greeks rather did geometric algebra! The algebra in their time was limited:

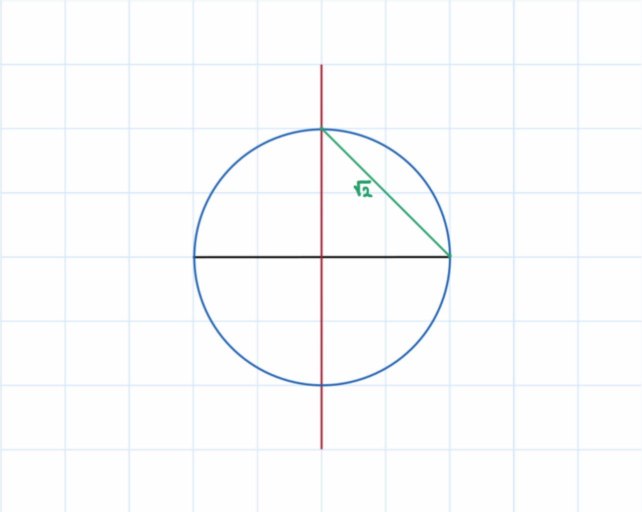
- All algebra needed to exist. There was no abstraction.
- Numbers were always natural. It was certainly *not* the case that the Greeks did not know about what we now call rational and irrational numbers. They simply did not consider them numbers.

Hence, the Greeks sought to express these quantities through geometric means!

Example 1: The geometric half



Example 2: The geometric $\sqrt{2}$



What distances can we find?

The Greeks wondered what geometric constructions could be done just with ruler and compass. The following proved particularly tricky:

- The duplication of the cube;
- Squaring the circle (i.e. finding a square with the area of a circle);
- Trisecting an angle.

It is only with modern mathematics that we are able to prove that these constructions are truly impossible with compass and ruler.

Duplication of the cube

In the fourth century BC, Menaechmus discovered the conic sections (ellipses, parabolas, hyperbolas).[1]

These provided a method to duplicate the cube. In modern algebraic notation, this would be the intersection of the parabola $y = x^2$ with the hyperbola $xy = 2$.

[1] Image taken from: Mathematics and Its History (A Concise Edition) by John Stillwell. Springer, 2020.

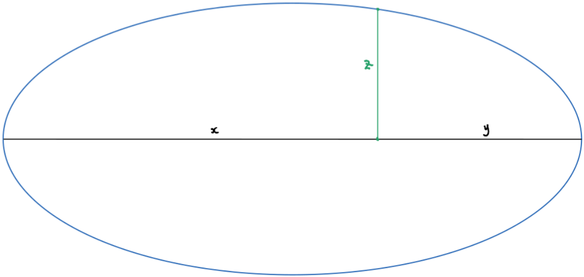
The future of Greek geometry

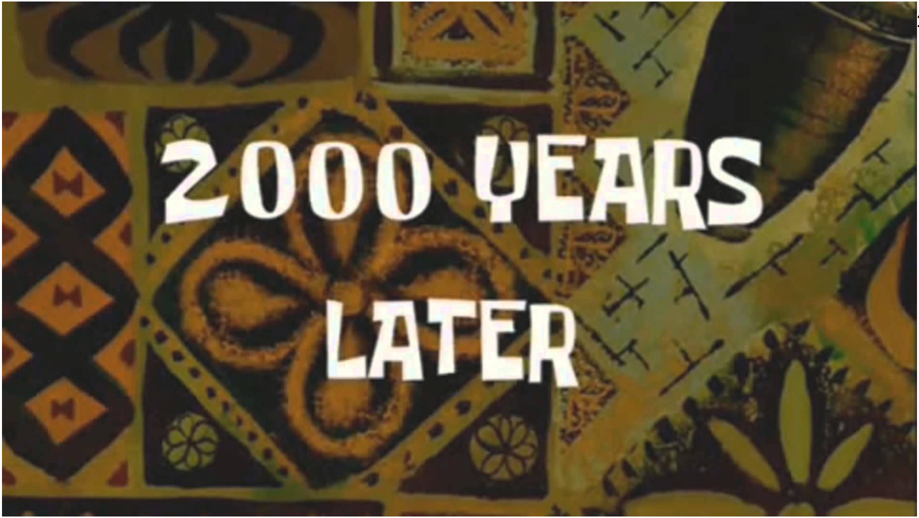
We still use the method of deductive mathematical reasoning that was introduced by the Greeks. However, Greek geometry generally did not stand the test of time, for the following reasons:

- Greek ‘geometric algebra’ was too weak and difficult.
- The Greek proofs were better suited for oral explanation.

Lack of coordinates

A whole other cause of problems in the method of the Greeks was the lack of a good coordinate system. Here is an example of the type of coordinates that Archimedes used:





[2]

[2]Image taken from: Spongebob, Nickelodeon. Season 1, Episode 14a: SB-129.

Creation (1630-1700)

The creators

The creation of algebraic geometry is simultaneously attributed to both Fermat (1629) and Descartes (1637). Two ingredients were crucial:

- A good coordinate system to effectively describe or draw functions or curves in.
- A good system of algebraic manipulation, to analyse and control these functions and curves.

Pierre de Fermat (1601-1665)

René Descartes (1596-1650)

A first classification

A very important theme throughout the history of algebraic geometry is classification. An important early classification of algebraic curves is by their degree:

- Curves of degree one are straight lines.
- Curves of degree two are the conic sections.
- What are the curves of degree three?

In 1695, Newton classified the cubics into seventy-two different species, missing six.

Some early limitations

Of course, not everything was immediately entirely general. Some things to keep in mind:

- Curves were still considered to be embedded into two-dimensonal space.
- Many curves were in fact not given by an algebraic equation. Hence the strong necessity for analytical methods.

The age of projective geometry (1639-1850)

Perspective

Perspective in art

'Definition'

Projective geometry is geometry which is independent from the point its observed from.

Projective geometry

Ideas of points at infinity were already present, but it was Desargues (1639) who set up a sound mathematical concept of projective geometry. It is worth noting that the first works of projective geometry did *not* rely on any algebraic methods.

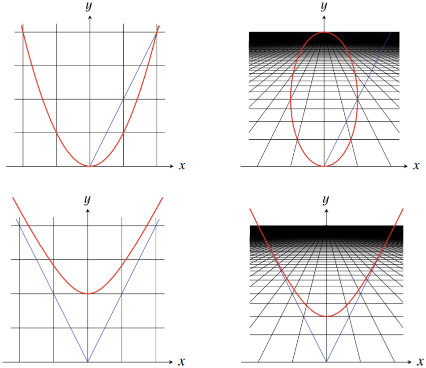
Girard Desargues (1591-1661)

This was a general tendency in the school of projective geometry: to distantiate as much from algebra as possible and rely more on heuristic principles. Foolish!

An algebraic foundation, namely that of homogeneous coordinates, was setup eventually by Möbius (1827) and Plücker (1830).

A simpler point of view

While more difficult to formalise, projective geometry simplifies a lot of constructions. For example, all conics are projectively equivalent![3]



Similarly, Newton's seventy-two cubics reduce to only three classes up to projective equivalence!

[3] Images taken from: Mathematics and Its History (A Concise Edition) by John Stillwell. Springer, 2020.

The new standard

However tough to implement, working in projective space, that is $\mathbb{P}^2(\mathbb{C})$ or $\mathbb{P}^3(\mathbb{C})$, became the geometric standard for the foreseeable future!