Introduction to zeta functions

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The Riemann zeta function is defined by a Dirichlet series:

DEFINITION

For all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, the Riemann zeta function is given by:

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}.$$

We can factorize the Riemann zeta function

PROPOSITION (EULER FACTORIZATION)

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

By gathering data on specific algebraic objects, one can generalize the concept of the Riemann zeta function:

$$\zeta_{\text{Obj}}(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \quad " = " \prod_{p \text{ "prime"}} \zeta_{p,\text{Obj}}(s).$$

Where

- a_n represents the algebraic data gathered on Obj
- "prime" is a set of elements depending on Obj
- $\zeta_{p,\text{Obj}}(s)$ are the local factors of the zeta function.

Let G be a finitely generated group. For all $n \in \mathbb{N}$ consider

$$a_n(G) = |\{H \leq G \mid [G:H] = n\}|.$$

Subgroup zeta function

One can define the subgroup zeta function by:

$$Z_G(s) = \sum_{n=1}^{+\infty} a_n(G) n^{-s}.$$

Example

$$Z_{\mathbb{Z}}(s) = \zeta(s) = \sum_{n=1}^{+\infty} n^{-s}.$$

Let G be a rigid group and let $\mathrm{Irr}(G)$ be the set of irreducible representations of G, up to equivalence. For all $n \in \mathbb{N}$, consider

$$r_n(G) = \left|\{\rho \in \mathrm{Irr}(G) \mid \dim \rho = n\}\right| .$$

Representation zeta function

One can define the representation zeta function by:

$$\zeta_G^{\mathrm{irr}}(s) = \sum_{n=1}^{+\infty} r_n(G) n^{-s}$$

Example

The representation zeta function of $\mathrm{SL}_2(\mathbb{F}_q)$ is

$$\zeta^{\mathrm{irr}}(s) = 1 + q^{-s} + \tfrac{q-3}{2}(q+1)^{-s} + \tfrac{q-1}{2}(q-1)^{-s} + 2\left(\tfrac{q+1}{2}\right)^{-s} + 2\left(\tfrac{q-1}{2}\right)^{-s}.$$

DEFINITION (POLYNOMIAL SUBGROUP GROWTH)

A finitely generated group G has polynomial subgroup growth (PSG) if the sequence $(s_N(G))_{N\in\mathbb{N}}$ with

$$\forall N \in \mathbb{N},\ s_N(G) = \sum_{i=1}^{N} a_i(G)$$

is bounded by a polynomial in N.

THEOREM

A finitely generated group G has (PSG) if and only if the subgroup zeta function Z_G converges.

Definition (Polynomial representation growth)

A rigid group G has polynomial representation growth (PRG) if the sequence $(R_N(G))_{N\in\mathbb{N}}$ with

$$\forall N \in \mathbb{N},\ R_N(G) = \sum_{i=1}^{N} r_i(G)$$

is bounded by a polynomial in N.

Theorem

A rigid group G has (PRG) if and only if the representation zeta function ζ_G^{irr} converges.

We define the abscissa of convergence $\alpha(G)$ of zeta function ζ_G as the infimum of all $\alpha \in \mathbb{R}$ such that ζ_G converges on the right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$.

Corollary

The abscissa of convergence of Z_G (resp. ζ_G^{irr}) is finite if and only if G has (PSG) (resp. (PRG)).

Recall the sequence $(R_N)_{N\in\mathbb{N}}$ defined by:

$$\forall\, N \in \mathbb{N},\ R_N(G) = \sum_{i=1}^{N} r_i(G).$$

If the growth sequence $(R_N)_{N\in\mathbb{N}}$ is unbounded, then

$$\alpha^{\text{rep}}(G) = \limsup_{N\to\infty} \frac{\log R_N(G)}{\log N}.$$

gives the polynomial degree of growth, namely

$$R_N(G) = O(N^{a^{\text{rep}}(G)+\varepsilon}) \text{ for all } \varepsilon > 0.$$

The same applies to the subgroup growth.

Let R be a complete discrete valuation ring of residue field $\mathbb{F}_q$. We have the following

Theorem (Jaikin-Zapirain)

The representation zeta function of $SL_2(R)$ is

$$\zeta^{irr}_{\mathrm{SL}_2(\mathbb{F}_q)} + \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \frac{(q-1)^2}{2}(q^2+q)^{-s}}{1-q^{1-s}}.$$

From this result, one can compute the abscissa of convergence:

$$\alpha^{\mathrm{rep}}\left(\mathrm{SL}_2(R)\right) = 1.$$

In fact, we know the following values for the abscissa of convergence of the special linear groups

n	$\alpha(\mathrm{SL}_n(R))$	
2	1	$= \frac{2}{n}$
3	$\frac{2}{3}$	
4	$\frac{1}{2}$	
≥ 5	≤ 2 and $\geq \frac{1}{15}$	

Open questions:

- General formula for the abscissa of convergence? How does it relate to the group?
- What is the first value of n for which $\frac{2}{n}$ fails?
- Does the sequence $\alpha(\mathrm{SL}_n(R))$ converge?

Let k a global field and let V be the set of all places of k. Consider S a subset of V and let

$$\mathcal{O}_S = \{x \in k \mid v(x) \geq 0 \text{ for all } v \notin S\}.$$

Then for G a k-algebric group, consider $\Gamma = G(k) \cap \mathrm{GL}_n(\mathcal{O}_S)$.

DEFINITION (PRINCIPAL S-CONGRUENCE SUBGROUP)

For an ideal $\mathfrak{a}$ of $\mathcal{O}_S$, we call principal S-congruence subgroup of level $\mathfrak{a}$ the group

$$\Gamma_{\mathfrak{a}} = \Gamma \cap \mathrm{GL}_n(\mathcal{O}_S, \mathfrak{a})$$

where $\mathrm{GL}_n(\mathcal{O}_S, \mathfrak{a})$ is the subgroup of $\mathrm{GL}_n(\mathcal{O}_S)$ consisting of matices congruent to the identity modulo $\mathfrak{a}$.

CONGRUENCE SUBGROUP PROBLEM

Is every subgroup of finite index of Γ an S-congruence subgroup?

Theorem (Lubotzky, Martin)

Let Γ be a "nice" arithmetic group. Then Γ has (PRG) is and only if Γ has the congruence subgroup property.

Theorem (Larsen, Lubotzky)

Let Γ be a "nice" arithmetic group with (CSP). Then Γ admits an Euler decomposition.

Example

For all $n \geq 3$, we have

$$\zeta^{\mathrm{irr}}_{\mathrm{SL}_n(\mathbb{Z})}(s) = \zeta^{\mathrm{irr}}_{\mathrm{SL}_n(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta^{\mathrm{irr}}_{\mathrm{SL}_n(\mathbb{Z}_p)}(s).$$