# Chow you doin'?
# GRK Retreat 2025

Jens Hornbostel, Matthias Wendt
Pengcheng Zhang, Alexander Ziegler

October 20, 2025

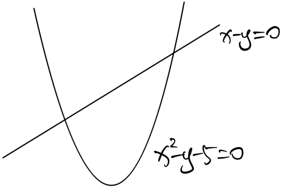
Heinrich Heine Universität Düsseldorf
BERGISCHE UNIVERSITÄT WUPPERTAL
RUHR UNIVERSITÄT BOCHUM

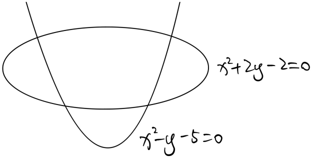
# Set-up

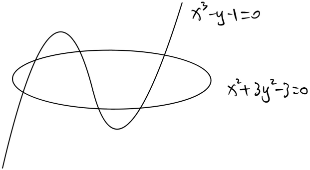
In this mini-course, we work entirely over $\mathbb{C}$. To avoid some technical details, we only work with varieties instead of schemes. All varieties considered here are irreducible and smooth, and equipped with Zariski topology.

The main reference is *3264 & All That, David Eisenbud & Joe Harris*.

# Beginning of Intersection Theory



# Bézout's Theorem

## Theorem (for plane curves)

*Let* $C_1, C_2$ *be two curves of degree* $d$ *and* $e$ *that intersect generically transversely in* $\mathbb{P}^2$. *Then they intersect at exactly* $d \cdot e$ *points.*

## Theorem (for general varieties)

*Let* $X_1, \ldots, X_k$ *be subvarieties of* $\mathbb{P}^n$ *of codimensions* $c_1, \ldots, c_k$ *with* $\sum_{i=1}^k c_i \leq n$. *If the* $X_i$ *intersect generically transversely, then*

$$\deg(X_1 \cap \cdots \cap X_k) = \prod_{i=1}^k \deg X_i.$$

**Definition**

*Let $X$ be a variety. The free Abelian group generated by all irreducible subvarieties of $X$ is called the group of cycles on $X$, denoted by $Z(X)$.*

The group $Z(X)$ is graded by dimension, i.e. $Z(X) = \bigoplus_{k=0}^{\dim X} Z_k(X)$, where each $Z_k(X)$ is the group of $k$-cycles on $X$.

A $k$-cycle $\sum n_i Y_i \in Z_k(X)$ is a formal linear combination of irreducible subvarieties $Y_i$ of dimension $k$ with integral coefficients.

# Rational Maps and Rational Functions

### Definition

*Let $X$ and $Y$ be two varieties. A rational map $f$ from $X$ to $Y$ is an equivalence class of pairs $(U, f_U)$, where $U$ is an open subset of $X$, and $f_U$ is a morphism from $U$ to $Y$, and two pairs $(U, f_U)$ and $(V, f_V)$ are equivalent if $f_U|_{U\cap V} = f_V|_{U\cap V}$.*
*In particular, when $Y = \mathbb{A}^1$, $f$ is called a rational function on $X$.*

All rational functions on a variety $X$ form a field, called the function field of $X$, denoted by $\mathbb{C}(X)$.

# Order of Zeros and Poles

Let $X$ be a variety. Take any non-zero rational function $f \in \mathbb{C}(X)$. Krull's principal ideal theorem implies that any irreducible component of the vanishing locus of $f$ on $X$ has codimension 1, which is a cycle in $Z_{\dim X-1}(X)$.

## Example

*On a projective curve $X$ over $\mathbb{C}$, we can find a Laurent series $\sum_n c_n(z-z_0)^n$ of a non-zero meromorphic function $f$ at a point $z_0 \in X$. The order of $f$ at $p$ is $\min\{n|c_n \neq 0\}$.*

# Divisor of a Rational Function

More generally, the ring $\mathcal{O}_{V,X}$ of rational functions on a codimension 1 subvariety $V$ of $X$ is a discrete valuation ring, since $X$ is smooth. Any $f \in \mathcal{O}^*_{V,X}$ can be written in the form $ug^m$, where $u \in \mathcal{O}_{V,X}$ is a unit and $g \in \mathcal{O}_{V,X}$ is a uniformizer of the unique maximal ideal of $\mathcal{O}_{V,X}$.

We say $m$ is the order of vanishing of $f$ along $V$ and set $\mathrm{ord}_V(f) = m$. Any function $r \in \mathcal{O}^*_X$ can be written as $r = r_1/r_2$, where $r_1, r_2 \in \mathcal{O}_{V,X}$. By defining $\mathrm{ord}_V(r) = \mathrm{ord}_V(r_1) - \mathrm{ord}_V(r_2)$, we obtain a group homomorphism from $\mathcal{O}^*_X$ to $\mathbb{Z}$.

# Divisor of a Rational Function

**Definition**

*Let $X$ be a variety. We define the divisor of $f \in \mathbb{C}(X)^*$ by*

$$\mathrm{Div}(f) = \sum_{\substack{V \subset X \\ \text{irreducible}}} \mathrm{ord}_V(f) V,$$

*where the sum ranges over all irreducible subvarieties $V \subset X$ of codimension 1.*

Similarly, for any $k+1$-dimensional subvariety $W \subset X$ and any $f \in \mathcal{O}_{W,X}$, we define the divisor of $f$ by

$$\mathrm{Div}(f) = \sum_{\substack{V \subset W \\ V \text{ irreducible}}} \mathrm{ord}_V(f) V,$$

the sum over all irreducible subvarieties $V \subset W$ of codimension 1.

# Divisor of a Rational Function

**Definition**

*A prime divisor on a variety $X$ is a closed subvariety of codimension 1. A Weil divisor is a formal sum of prime divisors. Any divisor that is equal to the divisor of a function $f \in \mathbb{C}(X)$ is called a principal divisor.*

# Rational Equivalence

## Definition

*Let $X$ be a variety and $Z(X)$ be the group of cycles. We say a $k$-cycle $W$ is rationally equivalent to 0 if there exist finitely many subvarieties $W_i$ of dimension $k+1$ and rational functions $f_i \in \mathcal{O}_{W_i,X}$ such that*

$$W = \sum_i \mathrm{Div}(f_i).$$

## Proposition

*Denote by $\mathrm{Rat}(X)$ the set of all cycles that are rationally equivalent to 0. Then $\mathrm{Rat}(\mathrm{X})$ is a subgroup of $Z(X)$.*

# Chow Group

**Definition**

*The Chow group of a variety $X$ is the quotient*

$$\mathrm{CH}^*(X) = Z(X)/\mathrm{Rat}(X).$$

*Denote by $[V]$ the equivalence class of a cycle $V$.*

Chow group is naturally graded by the dimension of cycles. From now on, we write $\mathrm{CH}^c(X) = \mathrm{CH}_{\dim X - c}$.

# Chow Ring

The next question is, how do we define a ring structure on $\mathrm{CH}^*(X)$?

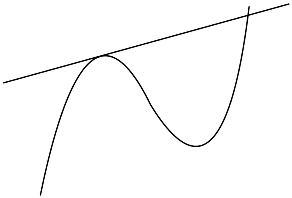
**Theorem**

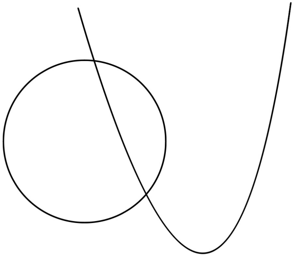
*Let $X$ be a variety. There exists a unique multiplication on $\mathrm{CH}^*(X)$ satisfying that if subvarieties $A, B \subset X$ are generically transverse, then $[A] \cdot [B] = [A \cap B]$.*
*This product makes $\mathrm{CH}^*(X)$ an associative, commutative ring graded by codimension.*

How can we be so sure that defining the product for generically transverse subvarieties is enough?

# Moving Lemma

What if this happens?





# Moving Lemma

**Theorem**

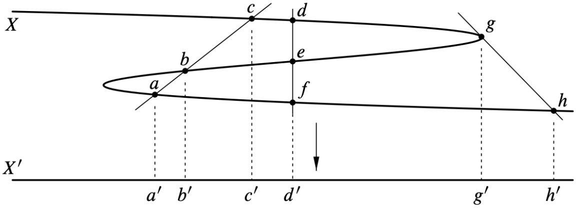
*Let $X$ be a variety.*

(a) *For every $[A], [B] \in \mathrm{CH}^*(X)$, there are generically transverse cycles $A', B' \in Z(X)$ with $[A'] = A$ and $[B'] = B$.*

(b) *The class $[A \cap B]$ is independent of the choice of $A'$ and $B'$.*

# Functoriality

$$a + b + c \sim d + e + f \sim 2g + h$$




$$a' + b' + c' \sim 3d' \sim 2g' + h'$$

# Functoriality

## Definition (Pushforward for cycles)

*Let $f : Y \to X$ be a proper morphism of varieties and let $A \subset Y$ be a subvariety.*

(a) *If* $\dim f(A) < \dim A$, *we set* $f_*(A) = 0$.

(b) *If* $\dim f(A) = \dim A$ *and* $f|_A$ *has degree* $n$, *we set* $f_*(A) = nA$.

(c) *We extend* $f_*$ *to all cycles in* $Z(Y)$ *by linearity.*

## Proposition

*The map* $f_* : Z(Y) \to Z(X)$ *defined above induces a morphism of groups* $f_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X)$ *for all* $k$.

# Functoriality

### Proposition (Pullback of cycles)

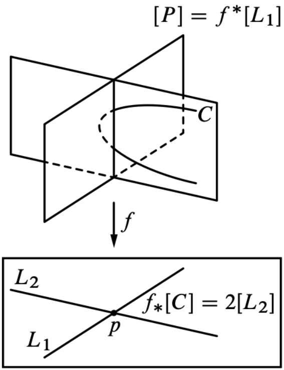
*Let $f : Y \to X$ be a morphism of varieties. There is a unique map of groups $f^* : \mathrm{CH}^k(X) \to \mathrm{CH}^k(Y)$ such that $f^*([A]) = [f^{-1}(A)]$ holds for any subvariety $A \subset X$ generically transverse to $f$, which means the preimage $f^{-1}(A)$ is generically reduced and $\mathrm{codim}_X(A) = \mathrm{codim}_Y(f^{-1}(A))$.*

### Proposition (Push-pull formula)

*Let $f : Y \to X$, $[A] \in \mathrm{CH}^*(X)$, and $[B] \in \mathrm{CH}^*(Y)$. Then*

$$f_*(f^*([A]) \cdot [B]) = [A] \cdot f_*([B])$$

# Functoriality



# Mayer-Vietoris and Localization Sequence

## Proposition

*Let $X$ be a variety.*

(a) *(Mayer-Vietoris) If $X_1, X_2$ are closed subvarieties of $X$, then there is a right exact sequence on the level of Chow groups*

$$\mathrm{CH}^*(X_1 \cap X_2) \to \mathrm{CH}^*(X_1) \oplus \mathrm{CH}^*(X_2) \to \mathrm{CH}^*(X_1 \cup X_2) \to 0.$$

(b) *(Localization/Excision) If $Y \subset X$ is a closed subvariety and $U = X \setminus Y$, then there is a right exact sequence on the level of Chow groups*

$$\mathrm{CH}^*(Y) \to \mathrm{CH}^*(X) \to \mathrm{CH}^*(U) \to 0.$$

# Localization Sequence

## Proof of (b).

Let $i : Y \to X, j : U \to X$ be inclusions. They induce morphisms of groups $i_* : Z_k(Y) \to Z_k(X), j^* : Z_k(X) \to Z_k(U)$ for all $k$. It can be directly verified that the sequence

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \to 0$$

is exact. It suffices to prove that the sequence remains exact after taking quotients of each term by rational equivalence.
If $[A] \in \ker(j^*)$, then $j^*([A]) = \sum[\mathrm{Div}(f_i)]$ where $f_i \in \mathcal{O}^*_{W_i}$, $W_i$ subvarieties of $U$. Viewing $W_i$ as a subvariety $\bar{W}_i$ of $X$, we may write $j^*([A] - \sum[\mathrm{Div}(\bar{f}_i)]) = 0$ in $Z_k(U)$, where $\bar{f}_i \in \mathcal{O}_{\bar{W}_i}$ corresponds to $f_i \in \mathcal{O}_{W_i}$. There exists some $[B] \in Z_k(Y)$ such that $[A] - \sum[\mathrm{Div}(\bar{f}_i)] = i_*([B])$.

# Homotopy Invariance and Affine Space

**Theorem**

*Let $\mathcal{E} \to X$ be a vector bundle. The pullback induces an isomorphism $CH^*(X) \to CH^*(\mathcal{E})$.*

**Corollary**

$$CH^*(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n].$$

**Proof.**

$\mathbb{A}^n$ is a free vector bundle over $\mathrm{Spec}(\mathbb{C})$ and

$$CH^*(\mathrm{Spec}(\mathbb{C})) = \mathbb{Z} \cdot [pt].$$

. □

# Counting Points

## Proposition

*Let $X$ be a projective variety. By proper pushforward the morphism $X \to \mathrm{Spec}(\mathbb{C})$ induces a surjective map*

$$CH^{\dim X}(X) \to CH^0(\mathrm{Spec}(\mathbb{C})) \cong \mathbb{Z}.$$

## Proof.

Proper pushfroward maps $[pt] \mapsto [pt]$ and

$$CH^*(\mathrm{Spec}(\mathbb{C})) = \mathbb{Z} \cdot [pt].$$

□

## Proposition

$$CH^*(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}).$$

*where* $\zeta^i$ *is the codimension* $i$ *hyperplane class.*

## Proof.

Using the localization sequence

$$CH^*(\mathbb{P}^{n-1}) \to CH^*(\mathbb{P}^n) \to CH^*(\mathbb{A}^n) \to 0$$

is exact giving us $CH^i(\mathbb{P}^{n-1}) \to CH^{i+1}(\mathbb{P}^n)$ is surjective.It is injective up to degree $n-1$ by composition with the pullback along $\mathbb{P}^{n-1} \to \mathbb{P}^n$.It is injective in degree $n$ by counting points. □

# Bézout's Theorem

**Theorem**

*Let $C_1, C_2$ be two algebraic curves in $\mathbb{P}^2$ of degrees $d_1, d_2$ that intersect transversely. Then they intersect in $d_1 \cdot d_2$ many points.*

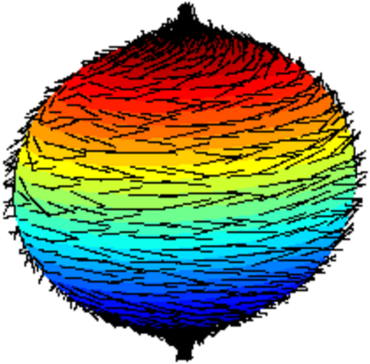
# Bézout's Theorem

**Theorem**

*Let $C_1, C_2$ be two algebraic curves in $\mathbb{P}^2$ of degrees $d_1, d_2$ that intersect transversely. Then they intersect in $d_1 \cdot d_2$ many points.*

**Proof.**

It suffices to show that the class of a degree $d$ curve $C$ is the $d$-fold multiple of the class of the line in $CH^*(\mathbb{P}^2)$. For this we pick a suitable line $L$ in $\mathbb{P}^2$ such that $C$ and $L$ intersect transversely. the intersection of $C$ and $L$ is given by a univariate degree $d$ polynomial which has $d$ roots. We can conclude $[C][L] = d[L]^2$ which implies $[C] = d[L]$. □



Public Domain, https://commons.wikimedia.org/wiki/File:Hairy_ball.png

# Divisors of Rational Sections

## Definition

*Let $\mathcal{L} \to X$ be a line bundle and $\tau$ a rational section. We can choose a covering $(U_i)_{i \in I}$ such that $\mathcal{L}$ restricts to a free line bundle on each $U_i$. In particular the $\tau|_{U_i}$ are rational functions that agree on intersections. We define $\mathrm{Div}(\tau)$ to be the gluing of the $\mathrm{Div}(\tau|_{U_i})$.*

## Proposition

*Let $\mathcal{L} \to X$ be a line bundle and $\sigma, \tau$ two rational sections. Their quotient $\sigma/\tau$ is a rational function and thus $\mathrm{Div}(\sigma/\tau)$ is a principal divisor.*

# Chern Classes of Line Bundles

## Definition

*Let $\mathcal{L} \to X$ be a line bundle. Define the Chern class of $\mathcal{L}$, denoted by $c_1(\mathcal{L}) \in CH^1(X)$ as the divisor of zeroes and poles $\mathrm{Div}(\tau)$ of a rational section $\tau$.*

## Proposition

*For two line bundles $\mathcal{L}_1, \mathcal{L} \to X$ we have $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$. In particular there is a group homomorphism $\mathrm{Pic}(X) \to CH^1(X)$.*

**Theorem**

*There is a unique way of assigning to a vector bundle $\mathcal{E} \to X$ of rank $r$ classes $c_i(\mathcal{E})$ such that*

(a) *$c_1(\mathcal{E})$ is given by the previous definition if $\mathcal{E}$ is a line bundle*

(b) *If the closed subset $D \subseteq X$ where global sections $\tau_1, \ldots, \tau_{r-i}$ are linearly dependent is of codimension $i$ then $c_i(\mathcal{E}) = [D]$.*

(c) *For a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ we have*

$$c_i(\mathcal{F}) = \prod_{j \leq i} c_j(\mathcal{E}) c_{i-j}(\mathcal{G}).$$

(d) *Chern classes commute with pullback.*

# Splitting Principle

**Theorem**

*Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

**Theorem**

*Every combing (section of the tangent bundle) of an algebraic hedgehog $\mathbb{P}^1$ admits two cowlicks (zeroes) of multiplicity 1 or one cowlick of multiplicity 2.*

**Proof.**

The tangent bundle of $\mathbb{P}^1$ is $\mathcal{O}(2)$. For a rational global sections $\tau$ of $\mathcal{O}(2)$ we have $\mathrm{Div}(\tau) = 2[pt]$ modulo principal divisors. □

# Enumerative Geometry

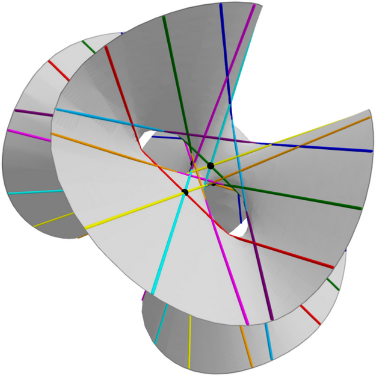
We want to count a class of geometric objects with certain conditions imposed for example lines on a smooth cubic surface.

- Find a suitable (smooth, projective) space $\mathcal{H}$ that parametrizes the geometric objects, for example a Grassmannian.
- Describe $CH^*(\mathcal{H})$.
- Find the class $c \in CH^*(\mathcal{H})$ of the locus of objects satisfying the conditions imposed, for example a Chern class.
- If the class is zero dimensional we can count the points in $c$.

# 27 Lines

## Theorem

*There are 27 lines on a smooth cubic surface.*



MIT license, https://commons.wikimedia.org/wiki/File:Clebsch_diagonal_cubic_surface.png

## Definition

*For $Z \subseteq X$ closed of codimension $n$, one can extend the localization sequence to a long exact sequence called motivic cohomology.*

$$\cdots \longrightarrow H^{2i-2,i}_{\mathrm{mot}}(X-Z) \to$$

$$\to H^{2(i-n)-1,i-n}_{\mathrm{mot}}(Z) \longrightarrow H^{2i-1,i}_{\mathrm{mot}}(X) \longrightarrow H^{2i-1,i}_{\mathrm{mot}}(X-Z) \to$$

$$\to CH^{i-n}(Z) \longrightarrow CH^{i}(X) \longrightarrow CH^{i}(X-Z) \longrightarrow 0$$

This fits into a larger motivic homotopy theory of simplicial sheaves over $\mathrm{Sm}_{\mathbb{C}}$ with respect to the Nisnevich topology.

# Motivic Obstruction Theory

There is a quadratic refinement of the Chow groups called Chow–Witt groups such that

$$\widetilde{CH}^{*}(\mathrm{Spec(k)}) = \widetilde{CH}^{0}(\mathrm{Spec(k)}) \cong \mathrm{GW}(k).$$

The Chow–Witt groups appear naturally as the habitat of obstruction classes for algebraic vector bundles over affine varieties. There is a natural map

$$\widetilde{CH}^{i}(X) \to CH^{i}(X)$$

which maps Euler classes to top Chern classes. (We have ignored the twist by a graded line bundle $\mathcal{L} \to X$.)

# The Cycle Class Map

**Definition**

*There is homomorphism of rings* $CH^*(X) \to H^{2*}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z})$ *which is natural with respect to pullback and preserves Chern classes. It is called the cycle class map.*

It is in general very difficult to make statements about the injectivity and surjectivity of this homomorphism.

# The Hodge Conjecture

For a projective variety $X$, can you describe the image of the map

$$CH^*(X) \otimes \mathbb{Q} \to H^{2*}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})?$$