Rationalité à la Igusa

Definable sets in $\mathbb{Q}_p$

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First-order logic (in short)

Write a formula using

- logical symbols: $=$ $(\ )$ $\wedge$ $\vee$ $\neg$ $\rightarrow$
- variables: $v_1\ v_2 \ldots$ or $x\ y\ z \ldots$, quantifiers: $\forall\ \exists$
- symbols from a fixed language, example: $+$ $<$ 0

$$\rightarrow 0 + ($$

BAD

$$\forall x (y < x \rightarrow \neg y = x)$$

$$\exists y\ x = y + y$$

VERY GOOD!!

Definable sets

Fix a language. Consider a first-order formula $\varphi(\overline{x})$ in this language, where $\overline{x} = (x_1, \ldots, x_n)$ are the free variables of φ.

Fix a first-order structure M in this language, that is, a set equipped with interpretations of symbols of my language.

The set $\varphi(M) := \left\{ \overline{a} \in M^{|\overline{x}|} \;\middle|\; \varphi(\overline{a}) \text{ is true in } M \right\}$ is a definable set.

Example: $\mathcal{L}_{rings} = \{0, 1, +, -, \times\}$, $\varphi(x)$: $\exists y (x = y^2)$

$\varphi(\mathbb{C}) = \mathbb{C}$

$\varphi(\mathbb{R}) = \mathbb{R}_{\geqslant 0}$

$\varphi(\mathbb{Q}_p) =$ see Immi's talk

$\varphi(\mathbb{Z}) = \{0, 1, 4, 9, \ldots\}$

Geometry of definable sets

Fix a language $\mathcal{L}$ and a structure M.

Let D be the collection of definable sets in M. Then:

- Singletons are in D, graphs of relations and of functions of the language are in D, including the diagonal (graph of equality).
- D is closed under permutations of coordinates, finite intersections, finite unions, complement, cartesian product.
- D is closed under projection.

In fact, D is the smallest such.

If $\mathcal{L} = \mathcal{L}_{ring}$, then D contains exactly the boolean combinations of algebraic varieties (the constructible sets) and their projections.

Stop projecting

Fix a language $\mathcal{L}$ and a structure M.

M 'eliminates quantifiers': every formula is equivalent to a formula without quantifiers.

Equivalently: definable sets are constructible.

Classical results:

- You just need to eliminate one quantifier.
- You can do things 'structurally': $\varphi(\overline{x})$ is equivalent to a quantifier-free formula iff anytime $N \equiv M$, anytime $\overline{a} \in M$ and $\overline{b} \in N$ are such that $\langle \overline{a} \rangle \simeq \langle \overline{b} \rangle$, then $M \vDash \varphi(\overline{a})$ iff $N \vDash \varphi(\overline{b})$.
- This is a notion relative to a language. You can always move to a larger language where M eliminates quantifiers.

Definable sets in $\mathbb{C}$

Tarski: $\mathbb{C}$ eliminates quantifiers in $\mathcal{L}_{rings}$.

Example:
$\exists y(x_n y^n + \cdots + x_0 = 0) \leftrightarrow x_0 = 0 \vee x_1 \neq 0 \vee \cdots \vee x_n \neq 0.$

Consequences:

- Chevalley's theorem: projections of constructible sets are constructible.
- Nullstellensatz: Any proper ideal of $\mathbb{C}[X_1, \ldots, X_n]$ has a common 0.
- Strong minimality: Definable sets in dimension 1 are finite or cofinite.

Definable sets in $\mathbb{R}$

The set of squares in $\mathbb{R}$ (also known as $\mathbb{R}_{\geqslant 0}$) is definable but not constructible. Hence, $\mathbb{R}$ does not eliminate quantifiers in $\mathcal{L}_{rings}$.

Tarski again: $\mathbb{R}$ eliminates quantifiers in $\mathcal{L}_{orings} = \{0, 1, +, -\times, <\}$.

Example: $\exists y(x_2 y^2 + x_1 y + x_0 = 0) \leftrightarrow (x_1^2 - 4x_2x_0 \geqslant 0)$.

Consequences:

- Hilbert 17: $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a sum of squares iff

$$\forall a_1 \ldots \forall a_n,\ f(a_1, \ldots, a_n) \geqslant 0.$$

- o-minimality: Definable sets in dimension 1 are finite unions of intervals and points.

Definable sets in $\mathbb{Q}_p$

The set of squares in $\mathbb{Q}_p$ is definable but not constructible. Hence, $\mathbb{Q}_p$ does not eliminate quantifiers in $\mathcal{L}_{rings}$. The same is true for the set of cubes or of n-power.

Macintyre: $\mathbb{Q}_p$ eliminates quantifiers in $\mathcal{L}_{Mac} = \{0, 1, +, -\times, (P_n)_{n \geqslant 2}\}$, where $P_n(x)$ holds iff $\exists y (x = y^n)$.

Note: $\mathbb{Z}_p$ is a definable set: $x \in \mathbb{Z}_p$ iff $\exists y (1 + px^2 = y^2)$. We then have $\mathbb{Z} \simeq \mathbb{Q}_p^\times / \mathbb{Z}_p^\times$, that is, $\mathbb{Z}$ is 'interpretable' in $\mathbb{Q}_p$.

Consequences:

- Definable sets in dimension 1 are finite (positive) Boolean combinations of P_n and points.
- Rationality results as explained by Immi: if $X \subseteq \mathbb{Z}_p^d \times \mathbb{Z}$ is definable, then $\sum_{m \geqslant 0} \mu(X_m) T^m$ is a rational function.
- $\mathbb{Q}_p$ is model-complete and decidable.