THE PROBABILISTIC ZETA FUNCTION

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Let G be a finitely generated profinite group, then we can define a formal Dirichlet series $P_G(s)$ as follows:

$$P_G(s) = \sum_{n\in\mathbb{N}} \frac{a_n(G)}{n^s} \qquad \text{with } a_n(G) = \sum_{|G:H|=n} \mu_G(H)$$

where $\mu_G(H)$ denotes the Möbius function of the lattice of open subgroups of G, defined recursively by

$$\mu_G(H) = \begin{cases} 1, & \text{if } H = G \\ -\sum_{H<K\leq G} \mu_G(K), & \text{otherwise} \end{cases}$$

for any proper open subgroup H of G.

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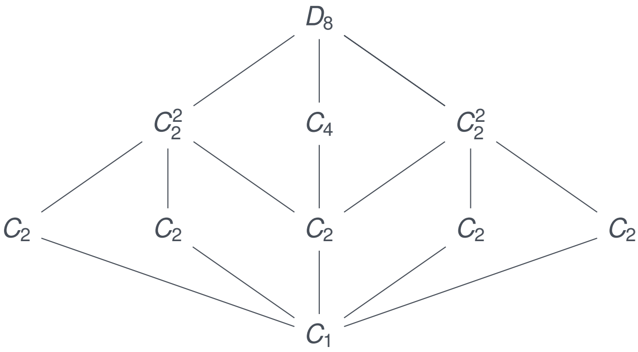
PROBABILISTIC ZETA FUNCTION

The formal inverse of the Dirichlet series $P_G(s)$ is called the probabilistic zeta function of G.

EXAMPLE $G = D_8$



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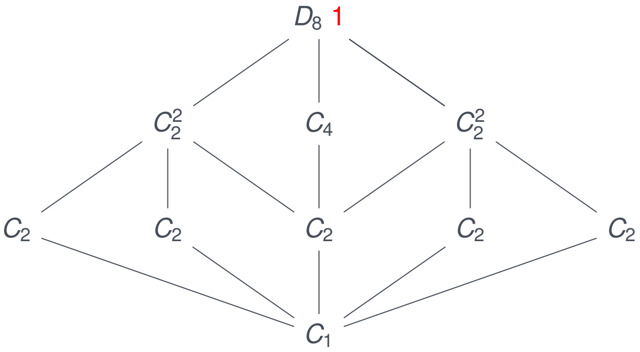

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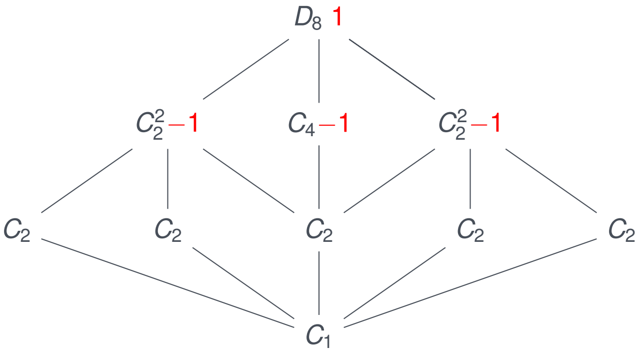

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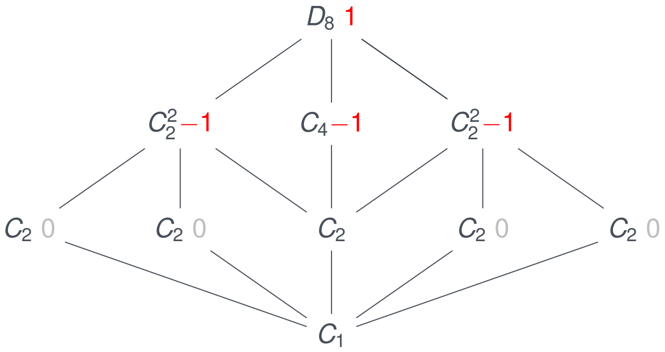

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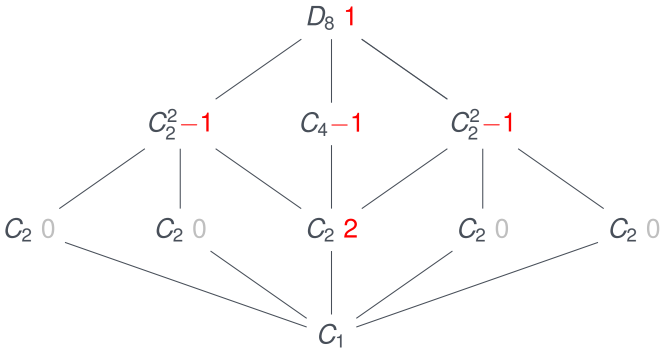

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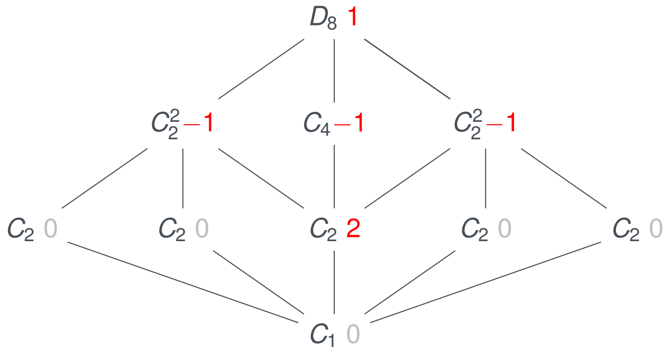

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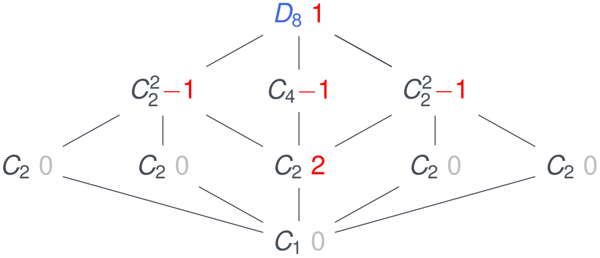

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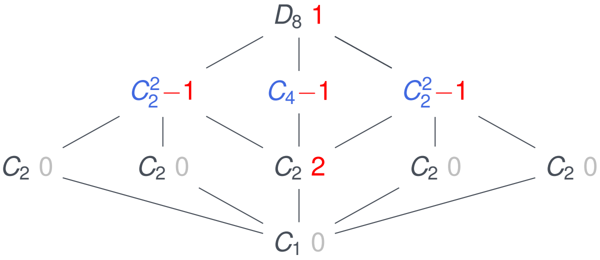

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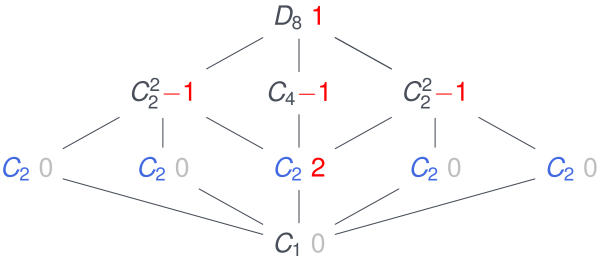

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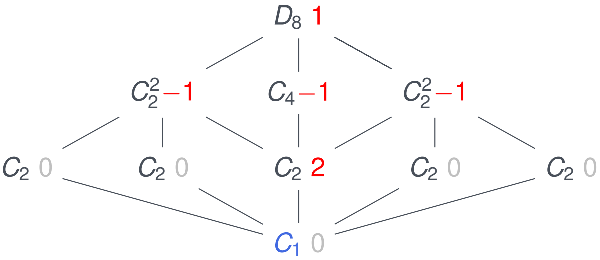

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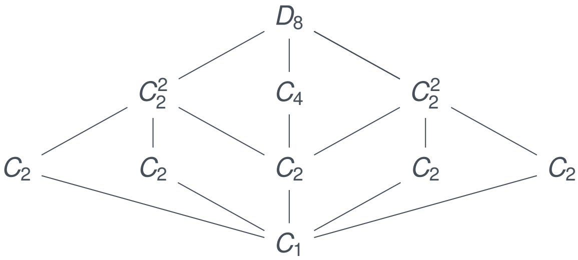

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$$P_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n(G)}{n^s} = 1 - \frac{3}{2^s} + \frac{2}{4^s}$$

Let H be a subgroup of G, then we have that $\mu_G(H) \neq 0 \Rightarrow H$ is an intersection of maximal subgroups of G.

We have that $P_G(s)$ depends only on the sublattice generated by the maximal subgroups of G and not on the complete subgroup lattice.

The probabilistic zeta function encodes information about the lattice generated by the maximal subgroups of G, just as the Riemann zeta function encodes information about the primes.

If $G = \widehat{\mathbb{Z}}$, then we have that

$$P_{\hat{\mathbb{Z}}}(s) = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right)^{-1} = (\zeta(s))^{-1}$$

where μ is the usual Möbius function.

So for $G = \hat{\mathbb{Z}}$ we have that the probabilistic zeta function coincides with the Riemann zeta function.

For a finite group G we get the following probabilistic interpretation:

THEOREM (P. HALL 1936)

Let G be a finite group, then for every positive integer t, we have that $P_G(t)$ is equal to the probability that t random elements generate the group G.

Since a profinite group G has a natural compact topology, it also has a Haar measure, and in particular there exists a unique normalized Haar measure ν on G.

So for a profinite group G we can define

$$Prob_G(t) = \nu(\{(g_1, \dots, g_t) \in G^t \mid \overline{\langle g_1 \dots, g_t \rangle} = G\})$$

the probability that a random t-uple generates G.

DEFINITION

A profinite group G is called PFG (positively finitely generated) if for some $t \in \mathbb{N}$ we get that $Prob_G(t) > 0$.

CONJECTURE (MANN 2004)

Let G be a PFG group, then the Dirichlet series $P_G(s)$ is absolutely convergent on some complex half-plane of $\mathbb{C}$ and, for sufficiently large $t \in \mathbb{N}$, we get $P_G(t) = Prob_G(t)$.

This conjecture has been proven to be true for some classes of profinite groups, for example for **finitely generated prosolvable groups** and groups that have **PSG**.

If G is a f.g. profinite group and $N \trianglelefteq G$ is a closed normal subgroup, then we can define the Dirichlet series $P_{G,N}(s)$ as follows:

$$P_{G,N}(s) = \sum_{n \in \mathbb{N}} \frac{b_n(G,N)}{n^s} \quad \text{with} \quad b_n(G,N) = \sum_{\substack{|G:H|=n \\ HN=G}} \mu_G(H)$$

and in particular we have that

$$P_G(s) = P_{G/N}(s) P_{G,N}(s)$$

If G is a finite group, then $P_{G,N}(t)$ is the conditional probability that a t-uple generates G, given that it generates G modulo N.

DEFINITION

Every finitely generated profinite group G has a chain $\{N_i\}_{i\in\mathbb{N}}$ of open normal subgroups

$$G = N_0 \trianglerighteq N_1 \trianglerighteq \dots \trianglerighteq N_i \trianglerighteq \dots$$

such that $\bigcap_{i\in\mathbb{N}} N_i = 1$ and each N_i/N_{i+1} is a minimal normal subgroup of G/N_{i+1}.

Such a chain is called a chief series of G.

THEOREM (DETOMI–LUCCHINI)

Let G be finitely generated profinite group and $\{N_i\}_{i\in\mathbb{N}}$ a chief series of G.
Then by iterating the formula $P_G(s) = P_{G/N}(s)P_{G,N}(s)$ we can write $P_G(s)$ as an infinite formal product:

$$P_G(s) = \prod_{i\in\mathbb{N}} P_i(s)$$

where each $P_i(s) = P_{G/N_{i+1}, N_i/N_{i+1}}(s)$ is the finite Dirichlet series associated to the chief factor N_i/N_{i+1}.

In particular the factors that appear in the product are independent of the choice of chief series.

Some ways in which $P_G(s)$ can be used to obtain information on the structure of the group G:

- Let G be a f.g. profinite group, then G is prosolvable $\iff P_G(s)$ is multiplicative (i.e. $a_{mn}(G) = a_m(G)a_n(G)$);
- Let G be a finite group, then G is p-solvable $\iff P_G(s)$ is p-multiplicative (i.e. $a_{p^r m}(G) = a_{p^r}(G)a_m(G)$);
- Let G be a finite group, then G is perfect $\iff n$ divides $a_n(G)$ for all $n \in \mathbb{N}$;
- Let G be a f.g. prosolvable group, then $G/\mathrm{Frat}(G)$ is finite $\iff P_G(s)$ is a finite Dirichlet series.

Thank you for your attention :)