Filtered Techniques for Noncommutative Rings

GRK 2240 Retreat – Gong Talks

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Gradings for the Polynomial Rings

Setting $\deg(x) = 1$ and $\deg(y) = 1$, we can define a **degree** for every polynomial in $\mathbb{C}[x, y]$. For instance:

$$\deg(x^5 + x + 1) = 5 \qquad \text{and} \qquad \deg(x^4 y + y^2 + x^3) = 5$$

This ***decomposes*** the polynomial ring $\mathbb{C}[x, y]$ as

$$\cdots \oplus \underbrace{(\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y)}_{=\mathbb{C}[x,y]_1} \oplus \underbrace{(\mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2)}_{=\mathbb{C}[x,y]_2} \oplus \cdots$$

so that $\dim_{\mathbb{C}} \mathbb{C}[x, y]_d < \infty$ for each summand $\mathbb{C}[x, y]_d$. Elements of the summand $\mathbb{C}[x, y]_d$ are called **homogeneous of degree d**.

Such a decomposition is called a **(standard) grading**.

Can we replicate this with suitable ***noncommutative*** rings?

Let $A_1(\mathbb{C})$ be the **(first) Weyl algebra**. This is a noncommutative $\mathbb{C}$-algebra with two generators x and y which do ***not*** commute but instead satisfy

$$[x, y] := xy - yx = 1 \quad \Longleftrightarrow \quad yx = xy - 1$$

Filtrations for the Weyl Algebras

Trying to define a **degree** by setting $\deg(x) = 1$ and $\deg(y) = 1$ does not work: the element yx ***ought*** to be homogeneous of degree 2, but $xy - 1$ is ***not*** homogeneous with respect to these degrees.

So either we change the natural grading or we relax from gradings to **filtrations**. This means, that instead of considering a ***summand*** of elements homogeneous of degree ***exactly d***, we consider a ***subgroup*** $A_1(\mathbb{C})_d$ of elements of degree ***at most d***.

This filters the Weyl algebra $A_1(\mathbb{C})$ as

$$\cdots \subset \underbrace{(\mathbb{C}{\cdot}1 \oplus \mathbb{C}{\cdot}x \oplus \mathbb{C}{\cdot}y)}_{=A_1(\mathbb{C})_1} \subset \underbrace{(\mathbb{C}{\cdot}1 \oplus \mathbb{C}{\cdot}x \oplus \mathbb{C}{\cdot}y \oplus \mathbb{C}{\cdot}x^2 \oplus \mathbb{C}{\cdot}xy \oplus \mathbb{C}{\cdot}y^2)}_{=A_1(\mathbb{C})_2} \subset \cdots$$

so that $\dim_{\mathbb{C}} A_1(\mathbb{C})_d < \infty$ for each subgroup $A_1(\mathbb{C})_d$.

The Punchline I

Note that $xy \in A_1(\mathbb{C})_2$, $1 \in A_1(\mathbb{C})_0$ and therefore $yx = xy - 1 \in A_1(\mathbb{C})_2$. In particular, x and y ***commute*** in the **associated graded ring** $\operatorname{gr} A_1(\mathbb{C})$:

$$\operatorname{gr} A_1(\mathbb{C}) = \cdots \oplus A_1(\mathbb{C})_1 / A_1(\mathbb{C})_0 \oplus A_1(\mathbb{C})_2 / A_1(\mathbb{C})_1 \oplus \cdots \cong \mathbb{C}[x, y]$$

It is possible to ***transfer*** properties from the associated graded ring $\operatorname{gr} A_1(\mathbb{C}) \cong \mathbb{C}[x, y]$ to the Weyl algebra $A_1(\mathbb{C})$.

The Punchline II

It is possible to ***transfer*** properties from the associated graded ring $\operatorname{gr} A_1(\mathbb{C}) \cong \mathbb{C}[x, y]$ to the Weyl algebra $A_1(\mathbb{C})$.

For instance, $\operatorname{gr} A_1(\mathbb{C}) \cong \mathbb{C}[x, y]$ is a ***noetherian regular*** commutative ring. This implies that $A_1(\mathbb{C})$ is a (left and right) ***noetherian*** (Auslander) ***regular*** noncommutative ring.