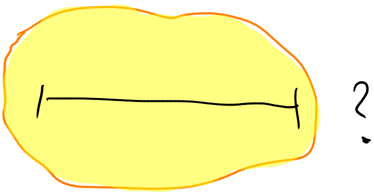
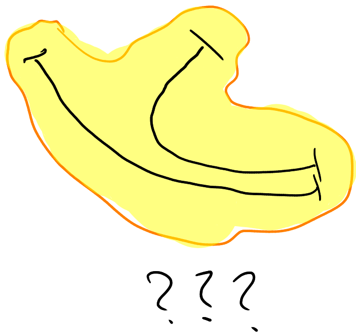
What is the length of a potato?

(Title stolen from notes by S. Schanuel)





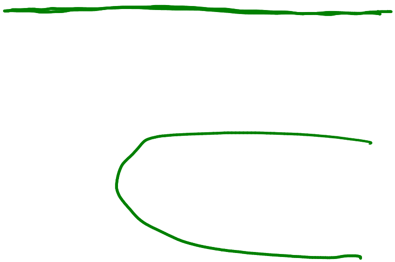
Goal of this talk: define this properly

Temporarily set dim(World) := 2

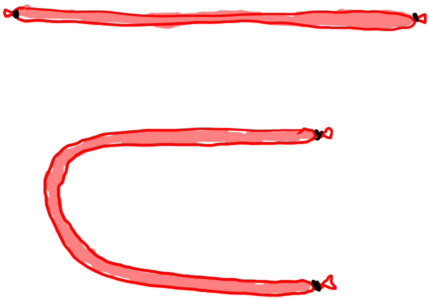
Remarks added afterwards:
The above-mentioned notes by Schanuel are: "What is the length of a potato? An introdution to geometric measure theory" (In: Categories in continuum physics (Buffalo, N.Y., 1982).) Those notes are very nice, but they describe something slightly different than I described in this talk. What I describe is call "Vitushkin variations"; a nice introduction to the entire topic is the book "Tame Geometry with Applications in Smooth Analysis" by Yomdin and Comte.

The length sausages

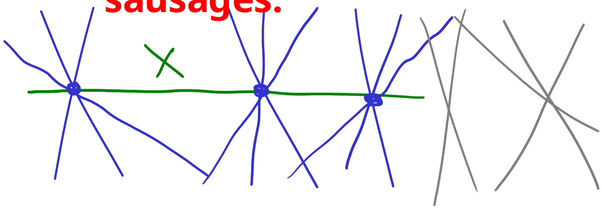
Curves:
Obvious notion of length:



Corresponding sausages:
Should have (almost) the same lengths:

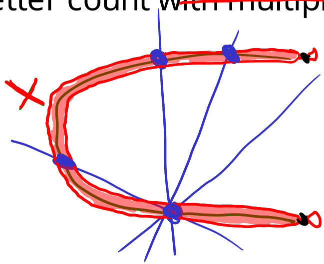


Measuring ~~curves:~~ sausages:



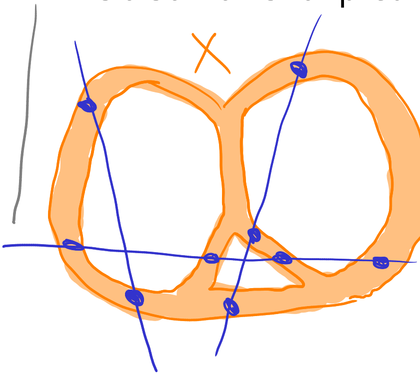
$$\mathrm{length}(X) = \mu\Big(\underbrace{\{L \text{ affine line} \mid L \cap X \neq \emptyset\}}_{\subset \ \text{space of affine lines}}\Big)$$

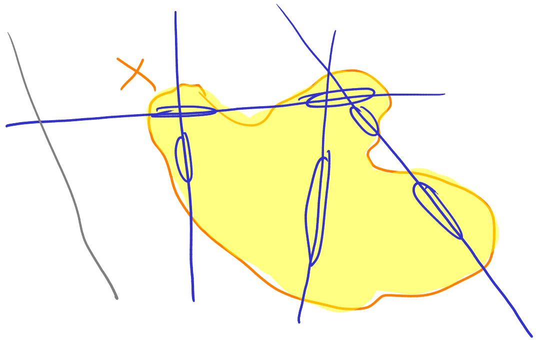
Better count ~~with multiplicities:~~ connected components



$$\mathrm{length}(X) = \int_{\substack{\text{space of} \\ \text{affine lines}}} \underbrace{\cancel{\#(X \cap L)}}_{\#\text{connected components of } X \cap L} \, dL$$

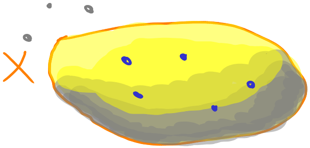
This also works for pretzels ... and potatos:



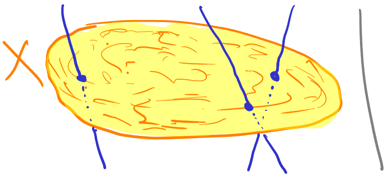


$$\text{length of potato} = \int_{\substack{\text{space of} \\ \text{affine lines}}} \#\text{conn. comps of } X \cap L \; dL$$

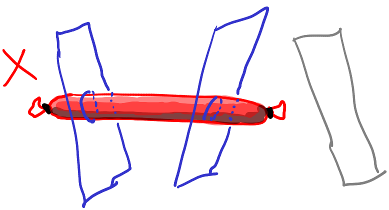
Other dimensions? (Ambient space, measuring dimension) dim(World) := 3



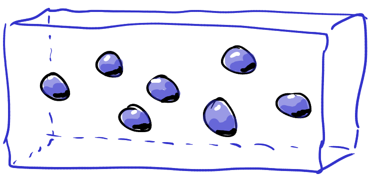
$$\mathrm{volume}(X) = \int_{\text{space of points}} \underbrace{\#\text{conn. comp of } X\cap L}_{\substack{\parallel \\ 1 \text{ if } L\in X \\ 0 \text{ if } L\notin X}} \, dL$$



$$\mathrm{area}(X) = \int_{\text{space of lines}} \#\text{conn. comp of } X\cap L \, dL$$



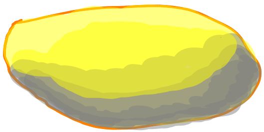
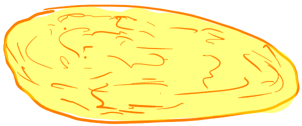
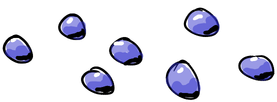
$$\mathrm{length}(X) = \int_{\text{space of planes}} \#\text{conn. comp of } X\cap L \, dL$$



$$\mathrm{number}(X) = \int_{\text{space of } \underbrace{\text{3-dim spaces}}_{\text{there is only one}}} \#\text{conn. comp of } X\cap L \, dL$$

How many bites does it take to eat it?

bite := subset contained in ball of radius 1/n

	food	#bites
	potato	$n^3 \cdot \text{volume}$
	pancake	$n^2 \cdot \text{area}$
	sausage	$n \cdot \text{length}$
	blue berries	number

Theorem: Any food item X can be eaten with

$$n^3 \cdot \text{volume}(X) + n^2 \cdot \text{area}(X) + n \cdot \text{length}(X) + \text{number}(X)$$

many bites of size 1/n.